VIETNAM NATIONAL UNIVERSITY, HA NOI

VNU UNIVERSITY OF SCIENCE

NGUYEN HONG SON

STABILITY AND ROBUST STABILITY OF DIFFERENTIAL-DIFFERENCE EQUATIONS WITH RESPECT TO STOCHASTIC PERTURBATIONS

Speciality: Probability Theory and Mathematical Statistics

Speciality Code: 9460112.02

SUMMARY OF A THESIS FOR THE DEGREE OF

DOCTOR OF PHILOSOPHY IN MATHEMATICS

HANOI – 2020

This work has been completed at VNU University of Science, Vietnam National University, Hanoi

Supervisors: **Assoc. Prof. Dr. Do Duc Thuan**
and **Assoc. Prof. Dr. Phan Viet Thu**

Referee 1: ....

Referee 2: ....

Referee 3: ....

To be defended at VNU University of Science, Vietnam National University, Hanoi :

At ....o'clock, ..., 2020

The thesis is publicly available at:

- National Library of Vietnam.
- VNU Library Information center, Hanoi.

Introduction

Stochastic modelling has come to play an important role in many branches of science and industry where more and more people have encountered stochastic differential equations as well as stochastic difference equations. The stochastic model can be used to solve a problem which evinces by accident, noise, etc.

This thesis is concerned with differential-algebraic equations (DAEs) subject to stochastic perturbations of the form

$$\begin{cases} Edx(t) = (Ax(t) + g(t))dt + f(t, x(t))dw(t), \\ x(t_0) = x_0, \end{cases} \tag{0.1}$$

where $E, A \in \mathbb{K}^{n\times n}$, the leading coefficient E is allowed to be a singular matrix and $w(t)$ is an m-dimensional Wiener process. While standard differential-algebraic equations (DAEs) without random noise are today standard mathematical models for dynamical systems in many application areas, such as multibody systems, electrical circuit simulation, control theory, fluid dynamics, and chemical engineering, the stochastic version is typically needed to model effects that do not arise deterministically. In fact, an accurate mathematical model of a dynamic system in electrical, mechanical, or control engineering often requires the consideration of stochastic elements. Electronic circuit systems or multibody mechanism systems with random noise are often modeled by stochastic differential algebraic equations (SDAEs), or sometime called stochastic implicit dynamic systems. These models have been studied recently. It is well known that, due to the fact that the dynamics of (0.1) are constrained, some extra difficulties appear in the analysis of stability as well as numerical treatments of SDAEs. These difficulties are typically characterized by index concepts. Note that, authors consider SDAEs only in the case of index-1.

As mentioned above, electronic circuit systems or multibody mechanism systems with random noise are often modeled by stochastic differential-algebraic equations (SDAEs), or sometimes called stochastic implicit dynamic systems. However, the advent of many modern-day sampled-data control systems has necessitated a study of stochastic discrete systems because they invariably include some stochastic elements that can only change at discrete instants of time. Examples of sampled data systems are digital computers, pulsed radar units, and coding units in most communication systems. These lead to stochastic implicit difference equations (SIDEs). They can also be obtained from SDAEs by some discretization methods. In the case of deterministic, an implicit difference equation (IDE) can be described in the form

$$E_n x(n+1) = A_n x(n) + q_n, n \in \mathbb{N}, \tag{0.2}$$

where $E_n, A_n \in \mathbb{R}^{d\times d}, x(n), q_n \in \mathbb{R}^d$ and E_n may be a singular matrix. IDEs are the generalization of regular explicit difference equations, which have been well investigated in the literature. They arise as mathematical models in various fields such as population dynamics, economics, systems and control theory, and numerical analysis. If $q_n = f(n)w_{n+1}$ is a random noise then we obtain a SIDE

$$E_n x(n+1) = A_n x(n) + f(n)w_{n+1}, n \in \mathbb{N}, \tag{0.3}$$

where w_{n+1} is a stochastic variable which is independent of the state $x(n)$. In this case, if E_n is the identity matrix then (0.3) becomes a stochastic difference equation which has attracted a good deal of attention from researchers in recent years. Unlike stochastic difference equations, the analysis of SIDEs is more complicated. Even the solvability analysis is not trivial.

On the other hand, in a lot of applications there is a frequently arising question, namely, how robust is a characteristic qualitative property of a system (e.g. stability) when the system comes under the effect of uncertain perturbations. The aspect of developing measures of stability robustness for linear uncertain systems with state-space description has received significant attention in system and control theory. These measures can be characterized by stability radius. The problem of evaluating and calculating this stability radius is of great importance, from both theoretical and practical points of view, and has attracted a lot of attention from researchers. For a systematic introduction to the topic, the interested readers are referred to the earlier work, which contains, along with rigorous theoretical developments, also an extensive literature review on the subject. It is remarkable that similar problems have been considered for many other types of linear dynamical systems, including time-varying and time-delay systems, implicit systems, positive systems, linear systems in infinite-dimensional spaces as well as linear systems with respect to stochastic perturbations.

On the basis of the above discussion, we have chosen the doctoral thesis research topic as "**Stability and robust stability of differential-difference equations with respect to stochastic perturbations"**. There arises a natural question whether one can define measures of stability robustness for DAEs respect to stochastic perturbations and, moreover, how to calculate these measures. To the best of our knowledge, such kind of questions has not been addressed so far in the literature, although different aspects of robust analysis for the stability of DAEs respect to deterministic perturbations has been studied already. The first purpose of the present thesis is to fill this gap. In the second chapter, we will study the consistency condition of random noise and define the index-ν concept for SDAEs. By using this index notion, we can establish the explicit expression of the solution and the variation of constants formula. After that, we shall establish the necessary and sufficient conditions for the exponential L^2-stability of SIDEs by using the method of Lyapunov functions which is well known for the stability theory of dynamic systems. As the main result in this chapter, we will establish the formula of the stability radius of DAEs with respect to stochastic perturbations. A problem, however, occurs in the case that the equation may not be solvable under stochastic perturbations, because then consistency conditions arise. To deal with this problem either a reformulation of the system has to be performed which characterizes the consistency conditions or the perturbations have to be further restricted.

In the third chapter, we are to perform the first investigation of SIDEs. The most important qualitative properties of SIDEs are solvability and stability. To study that, the index notion, which plays a key role in the qualitative theory of SIDEs, should be taken into consideration in the unique solvability and the stability analysis. Motivated by the index-1 concept for SDAEs and the index-ν concept in the second chapter, in this chapter we will derive the index-1 concept for time-varying SIDEs and the index-ν concept for SIDEs with constant coefficient matrices. By using this index notion, we can establish the explicit expression of solution, the variation of constants formula and the continuous dependence on initial condition of solution. On the other hand, the method of Lyapunov functions is well known for the stability theory of dynamic systems. By using this method, we shall establish the necessary conditions for the mean square stability of SIDEs. After that, characterizations of the mean square stability in the form of the quadratic Lyapunov equations are discovered.

The thesis is organized as follows.

- In the first chapter, we recall concepts of the stochastic process, the Drazin inverse, the index of a matrix pair, stochastic differential equations, and stochastic difference equations. We also mention some results on stability and stability for stochastic differential equations and stochastic difference equations.

- In the second chapter, the solvability of SDAEs is presented and the formula of the solution is derived. The mean square stability of SDAEs is studied and the formula of stability radii are established.

- In the third chapter, the solvability of SIDEs is investigated and the formula of the solution is provided. The mean square stability of SIDEs is derived by using the method of Lyapunov functions and the comparison theorem.

Chapter 1

Preliminary

1.1 Stochastic Processes

1.1.1. Basic notations of probability theory

Let $(\Omega, \mathcal{F}, P)$ be a probability space. A filtration is a family $\{\mathcal{F}_t\}_{t\geqslant 0}$ of increasing sub-σ-algebras of $\mathcal{F}$ (i.e. $\mathcal{F}_t \subset \mathcal{F}_s \subset \mathcal{F}$ for all $0 \leqslant t < s < \infty$). The filtration is said to be right continuous if $\mathcal{F}_t = \cap_{s>t}\mathcal{F}_s$ for all $t \geqslant 0$. When the probability space is complete, the filtration is said to satisfy the usual conditions if it is right continuous and $\mathcal{F}_0$ contains all $P-$null sets.

A family $\{X_t\}_{t\in\mathbb{R}}$ of $\mathbb{R}^d-$ valued random variables is called a stochastic process with parameter set (or index set) I and state space $\mathbb{R}^d$. The parameter set I is usually the halfline $\mathbb{R}_+ = [0, \infty)$, but it may also be an interval $[a, b]$, the non-negative integers or even subsets of $\mathbb{R}^d$. Note that for each fixed $t \in I$ we have a random variable $\Omega \ni \omega \to X_t(\omega) \in \mathbb{R}^d$.

On the other hand, for each fixed $\omega \in \Omega$ we have a function $I \ni t \to X_t(\omega) \in \mathbb{R}^d$, which is called a sample path of the process, and we shall write $X_\cdot(\omega)$ for the path. We often write a stochastic process $\{X_t\}_{t\geqslant 0}$ as $\{X_t\}, X_t$ or $X(t)$.

1.1.2. Stochastic integral

Definition 1.1.1. (Definition of Itô's integral) Let $f \in \mathcal{M}^2([a, b]; \mathbb{R})$. The Itô integral of f with respect to $\{w_t\}$ is defined by

$$\int_a^b f(t)dw_t = \lim_{n\to\infty} \int_a^b g_n(t)dw_t \;\text{ in } L^2(\Omega; \mathbb{R}), \tag{1.1}$$

where $\{g_n(t)\}$ is a sequence of simple processes such that

$$\lim_{n\to\infty} \mathbb{E} \int_a^b \|f(t) - g_n(t)\|^2 dt = 0. \tag{1.2}$$

1.1.3. Itô's formula

Theorem 1.1.2. *Let $x(t)$ be a $d-$dimensional Itô's process on $t \geqslant 0$ with the stochastic differential*

$$dx(t) = f(t)dt + g(t)dw(t),$$

where $f \in \mathcal{L}^1(\mathbb{R}_+; \mathbb{R}^d)$ and $g \in \mathcal{L}^2(\mathbb{R}_+; \mathbb{R}^{d\times m})$. Let $V \in C^{2,1}(\mathbb{R}^d \times \mathbb{R}_+; \mathbb{R})$. Then $V(x(t), t)$ is again an Itô's process with the stochastic differential given by

$$\begin{aligned} dV(x(t),t) =& \Big[V_t(x(t),t) + V_x(x(t),t)f(t) \\ &+ \frac{1}{2}\operatorname{Trace}\big(g^T(t)V_{xx}(x(t),t)g(t)\big)\Big]dt + V_x(x(t),t)g(t)dw(t) \;\; a.s. \end{aligned}$$

1.2 Stochastic differential equations

1.2.1. Definitions

Consider the d-dimensional stochastic differential equation of Itô type

$$dx(t) = f(x(t),t)dt + g(x(t),t)dw(t) \qquad \text{on } t_0 \leqslant t \leqslant T, \tag{1.3}$$

with initial value $x(t_0) = x_0$. By the definition of stochastic differential, this equation is equivalent to the following stochastic integral equation:

$$x(t) = x_0 + \int_{t_0}^t f(x(s),s)ds + \int_{t_0}^t g(x(s),s)dw(s) \qquad \text{on } t_0 \leqslant t \leqslant T. \tag{1.4}$$

Definition 1.2.1. An $\mathbb{R}^d-$valued stochastic process $\{x(t)\}_{t_0\leqslant t\leqslant T}$ is called a solution of equation (1.3) if it has the following properties:

(i) $\{x(t)\}$ is continuous and $\mathcal{F}_t-$adapted;

(ii) $\{f(x(t),t)\} \in L^1([t_0,T];\mathbb{R}^d)$ and $\{g(x(t),t)\} \in L^2([t_0,T];\mathbb{R}^{d\times m})$;

(iii) equation (1.4) holds for every $t \in [t_0, T]$ with probability 1.

A solution $\{x(t)\}$ is said to be unique if any other solution $\{\widehat{x}(t)\}$ is indistinguishable from $\{x(t)\}$, that is

$$P\{x(t) = \widehat{x}(t) \text{ for all } t_0 \leqslant t \leqslant T\} = 1.$$

1.2.2. Existence and uniqueness of solutions

Theorem 1.2.2. *Assume that there exist two positive constants $\widehat{K}$ and K such that*

(i) (Lipschitz condition) for all $x, y \in \mathbb{R}^d$ and $t \in [t_0, T]$

$$\|f(x,t) - f(y,t)\|^2 \vee \|g(x,t) - g(y,t)\|^2 \leqslant \widehat{K}\|x - y\|^2; \tag{1.5}$$

(ii) (Linear growth condition) for all $(x, t) \in \mathbb{R}^d \times [t_0, T]$

$$\|f(x,t)\|^2 \vee \|g(x,t)\|^2 \leqslant K(1 + \|x\|^2). \tag{1.6}$$

Then there exists a unique solution $x(t)$ to equation (1.3) *and the solution belongs to $\mathcal{M}^2([t_0, T]; \mathbb{R}^d)$.*

1.2.3. Stability of stochastic differential equations

Consider the following stochastic system

$$\begin{aligned} dx(t) &= Ax(t) + \sum_{j=1}^{N} D_j \Delta_j (Rx(t)) dw_j(t), \\ x(0) &= x_0. \end{aligned} \tag{1.7}$$

For $\Delta = (\Delta_1, ..., \Delta_N)$ we set $\|\Delta\| = [\sum_{j=1}^{N} ||\Delta_j||^2]^{1/2}$.

Definition 1.2.3. The stability radius of A with respect to the stochastic multi-perturbation structure $((D_j)_{j\in\mathbb{N}}, R)$ is

$$r_{\mathbb{K}}(A, (D_j)_{j\in\mathbb{N}}, R) = \inf\{||\Delta||; (1.7) \text{ is not } L^2 - \text{stable } \}.$$

Definition 1.2.4. The stability radius of A with respect to the stochastic multi-perturbation structure $((D_j)_{j\in\mathbb{N}}, R)$ is

$$r_K(A, (D_j)_{j\in\mathbb{N}}, R) = \inf\{||\Delta||; (1.7) \text{ is not exponentially } L^2 - \text{stable } \}$$

1.2.4. Stochastic differential algebraic equations

Consider the SDAEs of the type

$$Edx(t) + f(x(t), t)dt + G(x(t), t)dw(t) = 0, \tag{1.8}$$

where $f : \mathbb{R}^n \times [t_0, T] \to \mathbb{R}^n$ is a continuous vector-valued function of dimension $n, G : \mathbb{R}^n \times [t_0, T] \to \mathbb{R}^{n\times m}$ is an continuous $n \times m-$dimensional matrix-function, we understand it as a stochastic integral equation

$$Ex(s)|_{t_0}^t + \int_{t_0}^t f(x(s), s)ds + \int_{t_0}^t G(x(s), s)dw(s) = 0, \tag{1.9}$$

where the second integral is an Itô integral.

Definition 1.2.5. A strong solution of (1.9) is a process $x(\cdot) = (x(t))_{t\in[t_0,T]}$ with continuous sample paths that fulfills the following conditions:

- $x(\cdot)$ is adapted to the filtration $(\mathcal{F}_t)_{t\in[t_0,T]}$,
- $\int_{t_0}^t |f_i(x(s), s)|ds < \infty$ a.s., $\forall i = 1, ..., n,\ \forall t \in [t_0, T]$,
  $\int_{t_0}^t g_{ij}(x(s), s)dw(s) < \infty$ a.s., $\forall i = 1, ..., n,\ \forall j = 1, ..., m,\ \forall t \in [t_0, T]$,
- (1.9) holds a.s.

1.3 Stochastic difference equations

Now, let $\{\Omega, \mathcal{F}, P\}$ be a basic probability space, $\mathcal{F}_n \in \mathcal{F}, n \in \mathbb{N}$, be a family of σ-algebraic, $\mathbb{E}$ be an expectation, $\{w_n\} : w_n \in \mathbb{R}$ be a sequence of mutually independent $\mathcal{F}_n-$adapted random variables and independent on $\mathcal{F}_k, k < n$ satisfying $\mathbb{E}w_n = 0, \mathbb{E}w_n^2 = 1$ for all $n \in \mathbb{N}$. Consider the equation

$$x(n + 1) = A_n x(n) + R(n, x(n))w_{n+1},\ n \in \mathbb{N}, \tag{1.10}$$

with the initial condition

$$x(0) = x_0. \tag{1.11}$$

Here $R : \mathbb{N} \times \mathbb{R}^d \to \mathbb{R}^d$ is measurable.

Definition 1.3.1. The solution of (1.10) with the initial condition (1.11) is called:

- Mean square stable if for each $\varepsilon > 0$ there exists a $\delta > 0$ such that $\mathbb{E}\|x(n)\|^2 < \varepsilon, \forall n \in \mathbb{N}$, if $\mathbb{E}\|x_0\|^2 < \delta$.
- Asymptotically mean square stable if it is mean square stable and with $\mathbb{E}\|x_0\|^2 < \infty$ the solution $x(n)$ of (1.10) satisfies $\lim_{n\to\infty} \mathbb{E}\|x(n)\|^2 = 0$.

1.4 Index concepts

1.4.1. The concept of index-1 systems and theirs properties

Now, we introduce sub-spaces and matrices

$$S_n := \{z \in \mathbb{R}^d : A_n z \in \operatorname{im} E_n\},\ n \in \mathbb{N},$$
$$G_n := E_n - A_n T_n Q_n,\ \ P_n := I - Q_n,$$
$$\widetilde{Q}_{n-1} := -T_n Q_n G_n^{-1} A_n,\ \ \widetilde{P}_{n-1} := I - \widetilde{Q}_{n-1}.$$

Lemma 1.4.1. *The following assertions are equivalent*

a) $S_n \cap kerE_{n-1} = \{0\}$*;*

b) the matrices $G_n = E_n - A_n T_n Q_n$ *is non-singular;*

c) $\mathbb{R}^d = S_n \oplus kerE_{n-1}$.

We now consider a linear implicit difference equation

$$E_n x(n+1) = A_n x(n) + q_n,\ \ n \in \mathbb{N}, \tag{1.12}$$

and the homogeneous system associated with (1.12) is given by

$$E_n x(n+1) = A_n x(n),\ \ n \in \mathbb{N}, \tag{1.13}$$

where $E_n, A_n \in \mathbb{R}^{d\times d}, q_n \in \mathbb{R}^d$ and the matrix E_n may be singular.

Definition 1.4.2. The linear implicit difference equations (1.13) is said to be of index-1 tractable (index-1 for short) if for all $n \in \mathbb{N}$ the following conditions

(i) rank $E_n = r = constant$;

(ii) $\ker E_{n-1} \cap S_n = \{0\}$,

hold.

1.4.2. The Drazin inverse

Regular pairs (E, A) can be transformed to *Weierstraß-Kronecker canonical form*, i.e., there exist nonsingular matrices W, $T \in \mathbb{K}^{n\times n}$ such that

$$E = W \begin{bmatrix} I_r & 0 \\ 0 & N \end{bmatrix} T^{-1},\ A = W \begin{bmatrix} J & 0 \\ 0 & I_{n-r} \end{bmatrix} T^{-1}, \tag{1.14}$$

Definition 1.4.3. Consider a regular pair (E, A) with $E, A \in \mathbb{K}^{n\times n}$ in Weierstraß-Kronecker form (1.14). If $r < n$ and N has nilpotency index $\nu \in \{1, 2, ...\}$, i.e., $N^\nu = 0$, $N^i \neq 0$ for $i = 1, 2, ..., \nu - 1$, then ν is called the *index of the pair* (E, A) and we write $\text{Ind}(E, A) = \nu$. If $r = n$ then the pair has index $\nu = 0$.

Definition 1.4.4. Let $E \in \mathbb{K}^{n\times n}$ have $\nu = \text{Ind}E$. A matrix $X \in \mathbb{K}^{n\times n}$ satisfying

$$EX = XE, \tag{1.15a}$$

$$XEX = X, \tag{1.15b}$$

$$XE^{\nu+1} = E^\nu, \tag{1.15c}$$

is called a Drazin inverse of E.

Theorem 1.4.5. *Let $E \in \mathbb{K}^{n\times n}$ with $\nu = IndE$. There is one and only one decomposition*

$$E = \widetilde{C} + \widetilde{N} \tag{1.16}$$

with the properties

$$\widetilde{C}\widetilde{N} = \widetilde{N}\widetilde{C} = 0,\ \widetilde{N}^\nu = 0,\ \widetilde{N}^{\nu-1} \neq 0,\ Ind\widetilde{C} \leqslant 1. \tag{1.17}$$

In particular, the following statements hold:

$$\widetilde{C}^D\widetilde{N} = 0, \qquad \widetilde{N}\widetilde{C}^D = 0, \tag{1.18a}$$

$$E^D = \widetilde{C}^D, \tag{1.18b}$$

$$\widetilde{C}\widetilde{C}^D\widetilde{C} = \widetilde{C}, \tag{1.18c}$$

$$\widetilde{C}^D\widetilde{C} = E^DE, \tag{1.18d}$$

$$\widetilde{C} = EE^DE, \quad \widetilde{N} = E(I - E^DE). \tag{1.18e}$$

Theorem 1.4.6. *Let $E, A \in \mathbb{K}^{n\times n}$ satisfy $AE = EA$. Then we have*

$$EA^D = A^DE,\ E^DA = AE^D,\ E^DA^D = A^DE^D. \tag{1.19}$$

Moreover, if

$$kerE \cap kerA = \{0\} \tag{1.20}$$

then we have

$$(I - E^DE)A^DA = I - E^DE. \tag{1.21}$$

Chapter 2

Stability radii of differential-algebraic equations with respect to stochastic perturbations

2.1 Stochastic differential algebraic equations of index-ν

In this section, we consider the linear stochastic differential-algebraic equations with constant coefficients of the form

$$\begin{cases} Edx(t) = (Ax(t) + g(t))dt + f(t, x(t))dw(t), \\ x(t_0) = x_0, \end{cases} \tag{2.1}$$

where $E, A \in \mathbb{K}^{n\times n}$ are constant matrices, $g : [t_0, \infty) \to \mathbb{K}^n$ is a $(\nu - 1)$-times continuously differentiable vector-valued function, $w(t)$ is an m-dimensional Wiener process, $f : [t_0, \infty) \times \mathbb{K}^n \to \mathbb{K}^{n\times m}$ plays the role of a perturbation such that it is Lipschitz continuous in x, $f(t, x(t))$ is $\mathcal{F}$-adapted and $f(t, 0)$ is square integrable on $[t_0, T]$.

Definition 2.1.1. A function $x : [t_0, \infty) \times \Omega \to \mathbb{K}^n$ is a called solution of the initial value problem (2.1) if x is continuous and $\mathcal{F}$-adapted, $\int_{t_0}^{T} \|x(t)\|dt < \infty$ a.s., $\int_{t_0}^{T} \|f(t, x(t))\|^2 dt < \infty$ a.s. for $T > t_0$ and

$$Ex(t) = Ex_0 + \int_{t_0}^{t} (Ax(s) + g(s))ds + \int_{t_0}^{t} f(s, x(s))dw(s)$$

a.s. for all $t \in [t_0, \infty)$. The functions f, g and the initial condition x_0 is called consistent with (2.1) if the associated initial value problem has at least one solution. Equation (2.1) is called solvable if for every consistent f, g and x_0, the associated initial value problem has a solution.

2.1.1. Solvability of stochastic differential-algebraic equations

We first treat the special case where E and A commute, i.e.

$$EA = AE. \tag{2.2}$$

According to Theorem 1.4.5, we have decomposition $E = \widetilde{C}+\widetilde{N}$ with the properties of $\widetilde{C}$ and $\widetilde{N}$ as given there. We get the following lemma.

Lemma 2.1.2. *Equation* (2.1) *with property* (2.2) *is equivalent to the system*

$$\widetilde{C}dx_1(t) = Ax_1(t)dt + E^DEg(t)dt + E^DEf(t,x(t))dw(t), \tag{2.3a}$$

$$\widetilde{N}dx_2(t)=Ax_2(t)dt+(I-E^DE)g(t)dt + (I-E^DE)f(t,x(t))dw(t), \tag{2.3b}$$

where

$$x_1(t) = E^DEx(t), \;\; x_2(t) = (I-E^DE)x(t). \tag{2.4}$$

Moreover, equation (2.3a) *is equivalent to the stochastic differential equation*

$$dx_1(t) = E^DAx_1(t)dt + E^Dg(t)dt + E^Df(t,x(t))dw(t). \tag{2.5}$$

Proposition 2.1.3. *Let* $E, A \in \mathbb{K}^{n\times n}$ *satisfy* (2.2) *and* (1.20). *Then the consistent condition of the perturbation* f *for solvability of* (2.1) *is*

$$(I-E^DE)f = 0. \tag{2.6}$$

Moreover, the solution of equation (2.3b) *has only the form*

$$x_2(t) = -(I-E^DE)\sum_{i=0}^{\nu-1} A^D(A^D\widetilde{N})^i g^{(i)}(t), \text{ a.s. } \forall t \geqslant t_0, \tag{2.7}$$

and the consistent condition of g, x_0 *for solvability of* (2.1) *is*

$$(I-E^DE)\left(x_0 + \sum_{i=0}^{\nu-1} A^D(A^D\widetilde{N})^i g^{(i)}(t_0)\right) = 0.$$

Definition 2.1.4. The SDAE (2.1) is called tractable with index-ν (or for short, of index-ν) if

i) $\mathrm{Ind}(E,A) = \nu$,

ii) $(I-E^DE)f = 0$.

Remark 2.1.5. In the case $\nu = 1$ then the condition $(I - E^D E)f = 0$ is equivalent to $\operatorname{im} f \subset \operatorname{im} E$ and we return the notation of index-1

Theorem 2.1.6. *Asume that the SDAE* (2.1) *has index-ν and satisfies* (2.2). *Then, the solution of* (2.1) *is given by the formula*

$$\begin{aligned} x(t) =& e^{E^D A(t-t_0)} E^D E x_0 - (I - E^D E) \sum_{i=0}^{\nu-1} A^D (A^D \widetilde{N})^i g^{(i)}(t) \\ &+ \int_{t_0}^t e^{E^D A(t-s)} g(s) ds + \int_{t_0}^t e^{E^D A(t-s)} f(s, x(s)) dw. \end{aligned} \tag{2.8}$$

2.1.2. Stability for stochastic differential-algebraic equations

In this subsection, we study the L^2-stability and the exponential L^2-stability for SDAEs by using the method of Lyapunov functions. For defining the stability of zero solution, in equation (2.1) we assume that $g(t) = 0$ and $f(0, x) = 0$. Moreover, assume that there exist $a_1, a_2 > 0$ and a function $\gamma(t)$ such that $\|f(t, x)\| \leqslant \gamma(t)\|x\|$ and $\int_{t_0}^t \gamma^2(s) ds \leqslant a_1(t - t_0) + a_2$ for all $t \geqslant t_0$. Let us consider the equation

$$\begin{cases} E dx(t) = Ax(t)dt + f(t, x(t))dw(t), \\ x(t_0) = x_0, \end{cases} \tag{2.9}$$

where $E, A \in \mathbb{K}^{n\times n}$ are constant matrices and $w(t)$ be an m-dimensional Wiener process. For solvability of (2.9), by Proposition 2.1.3, the initial condition x_0 needs to satisty the consistent condition $(I - E^D E)x_0 = x_2(t_0) = 0$, or equivalently, $x_0 \in \operatorname{im}(E^D E)$.

Definition 2.1.7. Equation (2.9) is said to be L^2-stable if $\int_{t_0}^{\infty} \mathbb{E}(\|x(t, t_0, x_0)\|^2) dt < \infty$ for all $x_0 \in \operatorname{im}(E^D E)$. Equation (2.9) is said to be exponentially L^2-stable if there exist $\alpha, \beta > 0$ such that

$$\mathbb{E}\|x(t, t_0, x_0)\|^2 \leqslant \beta e^{-\alpha(t-t_0)} \|x_0\|^2, \tag{2.10}$$

for all $t \geqslant t_0 \geqslant 0$ and $x_0 \in \operatorname{Im}(E^D E)$.

Theorem 2.1.8. *Assume that the SDAE* (2.9) *has index-ν. Then equation* (2.9) *is exponentially L^2-stable if and only if there exists $\eta > 0$ such that*

$$\int_{t_0}^{\infty} \mathbb{E}(\|x(t, t_0, x_0)\|^2) dt \leqslant \eta \|x_0\|^2, \tag{2.11}$$

for all $t \geqslant t_0 \geqslant 0$ *and* $x_0 \in im(E^D E)$.

2.2 Stability radii for stochastic differential-algebraic equations with respect to stochastic perturbations

In this section, we will develop approach to investigate the robust stability of DAEs subject to stochastic perturbations. Consider the regular SDAEs

$$\begin{cases} Edx(t) = Ax(t)dt + C\Delta(Bx(t))dw(t), \\ x(t_0) = x_0, \end{cases} \tag{2.12}$$

where $E, A \in \mathbb{K}^{n\times n}$ are constant matrices, $B \in \mathbb{K}^{q\times n}$, $C \in \mathbb{K}^{n\times l}$ are structure matrices of perturbations, $w(t) \in \mathbb{R}^m, t \geqslant t_0 \geqslant 0$ are an m-dimensional Wiener process and x_0 is independent of $w(t), t \geqslant t_0 \geqslant 0$ and the disturbance operator $\Delta : \mathbb{K}^q \to \mathbb{K}^l$ is Lipschitz continuous with $\Delta(0) = 0$. The equation

$$Edx(t) = Ax(t)dt \tag{2.13}$$

is called the deterministic part of (2.12). Assume that $\sigma(E, A) \subset \mathbb{C}_-$, or equivalently, equation (2.13) is exponentially stable.

It is already known for the case of perturbed DAEs,that it is necessary to restrict the perturbations in order to get a meaningful concept of the structured stability radius, since a DAE system may lose its regularity, solvability and/or stability under infinitesimal perturbations. We therefore introduce the allowable stochastic perturbations which the consistency condition (3.18) is satisfied, i.e.,

$$(I - E^D E)C = 0. \tag{2.14}$$

Definition 2.2.1. Assume that condition (2.14) holds. Then, the L^2-stability radius and the exponential L^2-stability radius of the exponentially stable equation (2.13) with respect to the stochastic perturbation in the form of (2.12) are defined by

$$r^s_{\mathbb{K}}(E, A; C, B) = \inf\{\|\Delta\|; (2.12) \text{ is not } L^2 - \text{stable}\},$$
$$r^{es}_{\mathbb{K}}(E, A; C, B) = \inf\{\|\Delta\|; (2.12) \text{ is not exponentially } L^2 - \text{stable}\}.$$

By Theorem 2.1.6 and the consistent initial condition $E^D E x_0 = x_0$, the solution of (2.12) satisfies the equation

$$x(t) = e^{E^D A(t-t_0)} x_0 + \int_{t_0}^{t} e^{E^D A(t-s)} E^D C \Delta(Bx(s)) dw(s). \tag{2.15}$$

Let

$$\mathcal{V} = L^2[[t_0, \infty), L_2(\Omega, \mathbb{K}^{l\times m})], \mathcal{H}_0 = L^2[[t_0, \infty), L_2(\Omega, \mathbb{K}^{n\times n})],$$
$$\mathcal{H} = L^2[[t_0, \infty), L_2(\Omega, \mathbb{K}^{q\times n})].$$

The spaces $\mathcal{V}, \mathcal{H}_0, \mathcal{H}$ are equipped by the inner product $\langle \cdot, \cdot \rangle$ as follows

$$\langle u(\cdot), v(\cdot)\rangle = \int_{t_0}^{\infty} \mathbb{E}\langle u(t), v(t)\rangle dt = \int_{t_0}^{\infty} \mathbb{E}\,\mathrm{Trace}(v(t)^* u(t)) dt.$$

With this inner product, $\mathcal{V}, \mathcal{H}_0, \mathcal{H}$ become the Hilbert spaces. We now define the operators $\mathbb{M} : \mathcal{V} \longrightarrow \mathcal{H}_0$ by

$$(\mathbb{M}v)(t) = \int_{t_0}^{t} e^{E^D A(t-s)} E^D C v(s) dw(s), \tag{2.16}$$

and $\mathbb{L} : \mathcal{V} \longrightarrow \mathcal{H}$ by

$$(\mathbb{L}v)(t) = B(\mathbb{M}v)(t). \tag{2.17}$$

Using Weierstraß-Kronecker canonical form for commutative matrix pair, we have

$$e^{E^D A(t-s)} E^D = T \begin{bmatrix} e^{J(t-s)} & 0 \\ 0 & 0 \end{bmatrix} T^{-1},$$

where $J \in \mathbb{K}^{r\times r}$ with $\sigma(J) = \sigma(E, A) \subset \mathbb{C}_-$. Therefore there exist $K, \alpha > 0$ such that

$$\left\| e^{E^D A(t-s)} E^D \right\| \leqslant K e^{-\alpha(t-s)}.$$

This implies that the operators $\mathbb{M}$ and $\mathbb{L}$ are bounded. Now, we derive an upper bound for the perturbation such that equation (2.12) preserve the exponential stability.

Define the matrix

$$P_\rho = P_\rho^* = \rho^2 \int_{t_0}^{\infty} E^{D*} e^{(E^D A)^*(t-t_0)} B^* B e^{E^D A(t-t_0)} E^D dt. \tag{2.18}$$

This is well-defined because $\|e^{E^D At} E^D\| \leqslant K e^{-\alpha t}$. Since E and A commute, it implies that $e^{E^D At} E^D = E^D e^{E^D At}$. Therefore P_ρ is a solution of the Liapunov equation

$$P_\rho E^D A + (E^D A)^* P_\rho + \rho^2 (BE^D)^* BE^D = 0, \tag{2.19}$$

and satisfies $P_\rho(I - E^D E) = 0$. We now derive a computable formula for $\|\mathbb{L}\|$ based on the matrix P_ρ.

Proposition 2.2.2. *Let P_ρ be defined in equation* (2.18). *Then we have*

$$\|\mathbb{L}\|^{-1} = \sup\{\rho > 0 : I_l - C^* P_\rho C \geqslant 0\}. \tag{2.20}$$

By using construction of the stochastic perturbation destroying stability in [**?**], we will construct a stochastic perturbation preserving stability with the norm near the stability radius to get the formula of these radii.

Theorem 2.2.3. *Assume that condition* (2.14) *holds. Then, the stability radii of the exponentially stable equation* (2.13) *with respect to the stochastic perturbation in the form of* (2.12) *is given by the formula*

$$r^s_{\mathbb{K}}(E, A; C, B) = r^{es}_{\mathbb{K}}(E, A; C, B) = \|\mathbb{L}\|^{-1}. \tag{2.21}$$

Example 2.2.4. Consider a DAE subject to stochastic perturbation:

$$EdX = AXdt + C\Delta(BX)dw, \tag{2.22}$$

where

$$E = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, A = \begin{bmatrix} -2 & 2 \\ 2 & -3 \end{bmatrix}, C = \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 & -1 \end{bmatrix}.$$

By Theorem 2.2.3, we obtain

$$r^s_{\mathbb{K}}(E, A; C, B) = r^{es}_{\mathbb{K}}(E, A; C, B) = r^s_{\mathbb{K}}(E_1, A_1; C_1, B) = r^{es}_{\mathbb{K}}(E_1, A_1; C_1, B) = \sqrt{2}.$$

Now, we construct two perturbations

$$\Delta_1(y) = \rho_1 \|y\| z_1, \Delta_2(y) = \rho_2 \|y\| z_2,$$

where $\rho_1 = 1.5392, z_1 = \begin{bmatrix} 0.866 \\ 0.5 \end{bmatrix}, \rho_2 = 1,412, z_2 = \begin{bmatrix} 0.866 \\ -0.5 \end{bmatrix}$. Then, it is easy to see that $\|\Delta_1\| = \rho_1 > \|\mathbb{L}\|^{-1} > \|\Delta_2\| = \rho_2$. With the perturbation Δ_1, equation (2.12) is exponentially L^2-unstable.With the perturbation Δ_2, equation (2.12) is exponentially L^2-stable.

Chapter 3

Stochastic implicit difference equations

3.1 Stochastic implicit difference equations of index-1

Let us consider the stochastic implicit difference equations (SIDEs)

$$E_n x(n+1) = A_n x(n) + R(n, x(n)) w_{n+1},\ n \in \mathbb{N}, \tag{3.1}$$

with the initial condition $x(0) = \widetilde{P}_{-1} x_0$, where $E_n,\ A_n \in \mathbb{R}^{d\times d}$ with rank $E_n = r < d$, the function $R : \mathbb{N} \times \mathbb{R}^d \to \mathbb{R}^d$ is measurable and $\{w_n\} : w_n \in \mathbb{R}$ is a sequence of mutually independent $\mathcal{F}_n$-adapted random variables, and independent on $\mathcal{F}_k, k < n$ satisfying $\mathbb{E} w_n = 0, \mathbb{E} w_n^2 = 1$ for all $n \in \mathbb{N}$. The homogeneous equation associated to (3.1) is

$$E_n x(n+1) = A_n x(n),\ n \in \mathbb{N}. \tag{3.2}$$

Now, we give a rigorous definition of solution of (3.1).

3.1.1. Solutions of stochastic implicit difference equations

Definition 3.1.1. A stochastic process $\{x(n)\}$ is said to be a solution of the SIDEs (3.1) if with probability 1, $x(n)$ satisfies (3.1) for all $n \in N$ and $x(n)$ is $\mathcal{F}_n-$ measurable.

Definition 3.1.2. The SIDE (3.1) called tractable with index-1 (or for short, of index-1) if

i) The deterministic part (3.2) of (3.1) is a linear IDE with index-1.

ii) $\operatorname{im} R(n, \cdot) \subset \operatorname{im} E_n$ for all $n \in \mathbb{N}$.

By using the above notion, we solve the problem of existence and uniqueness of solution of (3.1) in the following theorem.

Theorem 3.1.3. *If equation* (3.1) *is of index-1, then for any $n \in \mathbb{N}$ and with the initial condition $x(0) = \widetilde{P}_{-1}x_0$, it admits a unique solution $x(n)$ which given by the formula*

$$x(n) = \widetilde{P}_{n-1}u(n), \tag{3.3}$$

where $\{u_n\}$ is a sequence of $\mathcal{F}_n$-adapted random variables defined by the equation

$$u(n+1) = P_n G_n^{-1} A_n u(n) + P_n G_n^{-1} R(n, \widetilde{P}_{n-1}u(n))w_{n+1}, \ n \in \mathbb{N}.$$

3.1.2. The variation of constants formula for stochastic implicit difference equations

Now, to construct the variation of constants formula for equation (3.1), we need to define the Cauchy operator $\Phi(n,m)$ of the corresponding homogeneous equation (3.2). We have the following proposition.

Proposition 3.1.4. *Let $\Phi(n,m)$ be the Cauchy operator $\Phi(n,m)$ of equation* (3.2). *Then, we have*

i) $\Phi(m,m) = \widetilde{P}_{m-1}$;

ii) $\Phi(n,m)\Phi(m,k) = \Phi(n,k)$;

iii) $\Phi(n,m) = \prod_{k=m}^{n-1} \widetilde{P}_k G_k^{-1} B_k$.

Now we derive the variation of constants formula for the solution of equation (3.1) in the following theorem.

Theorem 3.1.5. *The unique solution of equation* (3.1) *can be expressed as*

$$x(n) = \Phi(n,m)P_{m-1}x(m) + \sum_{i=m}^{n-1} \Phi(n,i+1)P_i G_i^{-1} R(i,x(i))w_{i+1}, \tag{3.4}$$

where $\Phi(n,m)$ is the fundamental matrix of equation (3.2).

3.1.3. Dependence on the consistent initial condition of solution

Next, we consider the dependence on the consistent initial condition of solution of equation (3.1). Assume that $\mathbb{R}^d$ is endowed with the Euclidean norm and there exists $b_n > 0$ such that

$$\|R(n,x) - R(n,0)\| \leqslant b_n\|x\|, \text{ for all } x \in \mathbb{R}^d.$$

Let

$$\alpha := \sup_{0\leqslant n\leqslant \mathbb{N}} \left\{ \|P_n G_n^{-1} A_n\|^2 + 2b_n^2 \|P_n G_n^{-1}\|^2 \|\widetilde{P}_{n-1}\|^2 \right\},$$

$$\beta := \sup_{0\leqslant n\leqslant \mathbb{N}} \|P_n G_n^{-1} R(n,0)\|, \ \gamma := \sup_{0\leqslant n\leqslant \mathbb{N}} \|\widetilde{P}_{n-1}\|.$$

Theorem 3.1.6. *Assume that $x(n)$ be the unique solution of equation (3.1) and $\mathbb{E}\|P_{-1}x(0)\|^2 < \infty, n \in \mathbb{N}$. Then the following inequalities hold for all $0 \leqslant n \leqslant \mathbb{N}$*

$$\mathbb{E}\|x(n)\|^2 \leqslant \alpha^n \gamma^2 \mathbb{E}\|P_{-1}x(0)\|^2 + \frac{2\beta^2\gamma^2(\alpha^n - 1)}{\alpha - 1}. \tag{3.5}$$

3.2 Stability of stochastic implicit difference equations of index-1

In this section, we study stability of the SIDE (3.1) of index-1. It is well known that the method of Lyapunov functions is very useful to investigate stability of dynamic systems. Thus, we will use this method to derive characterizations of mean square stability for equation (3.1). First, we introduce the following stability notion which is generalized from Definition 1.3.1 for stochastic difference equations.

3.2.1. Stability of stochastic implicit difference equations

Definition 3.2.1. The trivial solution of equation (3.1) is called:

- Mean square stable if for any $\varepsilon > 0$ and there exists a $\delta > 0$ such that $\mathbb{E}\|x(n)\|^2 < \varepsilon, \forall n \in \mathbb{N}$, if $\mathbb{E}\|P_{-1}x(0)\|^2 < \delta$.
- Asymptotically mean square stable if it is mean square stable and with

$$\mathbb{E}\|P_{-1}x(0)\|^2 < \infty$$

  the solution $x(n)$ of (3.1) satisfies $\lim_{n\to\infty} \mathbb{E}\|x(n)\|^2 = 0$.

Theorem 3.2.2. *Assume that $\gamma_0 := \sup_{n\geqslant 0} \|\widetilde{P}_{n-1}\| < \infty$ and there exists a non-negative function $V_n = V(n, P_{n-1}x(n))$ which satisfies the conditions*

$$\mathbb{E}V(0, P_{-1}x(0)) \leqslant c_1 \mathbb{E}\|P_{-1}x(0)\|^2, \tag{3.6}$$

$$\mathbb{E}\Delta V_n \leqslant -c_2\mathbb{E}\|P_{n-1}x(n)\|^2, \quad n \in \mathbb{N}, \tag{3.7}$$

where c_1, c_2 *and* p *are positive constants. Then the trivial solution of equation* (3.1) *is asymptotically mean square stable.*

From Theorem 3.2.2, it follows that stability of SIDEs can be reduced to construction of appropriate Lyapunov functions. Now, we derive characterizations for stability of SIDEs in the form of the quadratic Lyapunov equations.

Theorem 3.2.3. *Assume that there exist* $a_n, b_n, c_2 > 0$ *such that* $||R(n,x)|| \leqslant b_n||x||$ *and the matrix equation*

$$A_n^T H_{n+1} A_n - E_{n-1}^T H_n E_{n-1} = -a_n^2 P_{n-1}^T P_{n-1} \tag{3.8}$$

has a nonnegative definite solution H_n *satisfying for all* $n \geqslant 0$

$$b_n^2 \|H_{n+1}\| \|\widetilde{P}_{n-1}\|^2 - a_n^2 \leqslant -c_2 < 0. \tag{3.9}$$

Then the trivial solution of (3.1) *is asymptotically mean square stable.*

In the rest of this subsection, we consider implicit difference equations with respect to linear stochastic perturbations. Let $R(n, x(n)) = Z_n x(n)$ with $Z_n \in \mathbb{R}^{d\times d}$, then equation (3.1) becomes

$$E_n x(n+1) = A_n x(n) + Z_n x(n) w_{n+1}, \ n \in \mathbb{N}, \tag{3.10}$$

with the consistent initial condition

$$x(0) = \widetilde{P}_{-1} x_0. \tag{3.11}$$

Theorem 3.2.4. *Assume that there exist* $c_2 > 0$ *and a nonnegative definite matrix* H_n *in* $\mathbb{R}^d$ *satisfying*

$$A_n^T H_{n+1} A_n - E_{n-1}^T H_n E_{n-1} + Z_n^T H_{n+1} Z_n = -c_2 P_{n-1}^T P_{n-1}. \tag{3.12}$$

Then the trivial solution of equation (3.10) *is asymptotically mean square stable. Moreover, if* $c_2 < 0$ *and equation* (3.12) *has a nonnegative definite solution* H_n *then the trivial solution of equation* (3.10) *is not asymptotically mean square stable.*

3.2.2. A comparison theorem for stability of linear stochastic implicit difference equations of index-1

Theorem 3.2.5. *Assume that* $K_1 := \sup_{n\geqslant 0}\|\widetilde{P}_{n-1}\| < \infty$. *Then if there exists a positive sequence* $\{\alpha_n\}$ *with* $K_2 := \sum_{n=0}^{\infty}\alpha_n < \infty$ *such that*

$$\|P_nG_n^{-1}A_n\|^2 + \|P_nG_n^{-1}Z_n\widetilde{P}_{n-1}\|^2 \leqslant 1+\alpha_n,\ \forall n\geqslant 0,$$

then equation (3.10) *is mean square stable. If there exists a positive sequence* $\{\beta_n\}$ *with* $\sum_{n=0}^{\infty}\beta_n = \infty$ *such that*

$$\|P_nG_n^{-1}A_n\|^2 + \|P_nG_n^{-1}Z_n\widetilde{P}_{n-1}\|^2 \leqslant 1-\beta_n,\ \forall n\geqslant 0,$$

then equation (3.10) *is asymptotically mean square stable.*

3.3 Stochastic singular difference equations of index-ν

In this section, we consider the linear stochastic singular difference equations (SSDEs) with constant coefficients of the form

$$\begin{cases} Ex(n+1) = Ax(n) + f(n,x(n))w_{n+1}, \\ x(n_0) = x_0, \end{cases} \tag{3.13}$$

where $E, A \in \mathbb{K}^{n\times n}$ are constant matrices, $f : \mathbb{N}\times\mathbb{K}^n \to \mathbb{K}^n$ plays the role of a perturbation.Without lost of generality assume that $f(n,x(n))$ is measurable

3.3.1. Solvability for stochastic singular difference equations of index-ν

First, we also treat the special case where E and A commute, i.e.

$$EA = AE. \tag{3.14}$$

According to Theorem 1.4.5, we have decomposition $E = \widetilde{C}+\widetilde{N}$ with the properties of $\widetilde{C}$ and $\widetilde{N}$ as given there. We get the following lemma.

Lemma 3.3.1. *Equation* (3.13) *with property* (2.2) *is equivalent to the system*

$$\widetilde{C}x_1(n+1) = Ax_1(n) + E^DEf(n,x(n))w_{n+1}, \tag{3.15a}$$

$$\widetilde{N}x_2(n+1) = Ax_2(n) + (I-E^DE)f(n,x(n))w_{n+1}, \tag{3.15b}$$

where

$$x_1(n) = E^D Ex(n), \quad x_2(n) = (I - E^D E)x(n). \tag{3.16}$$

Moreover, equation (3.15a) *is equivalent to the stochastic differential equation*

$$x_1(n+1) = E^D Ax_1(n) + E^D f(n, x(n))w_{n+1}. \tag{3.17}$$

Definition 3.3.2. A stochastic process $\{x(n)\}$ is said to be a solution of the SSDEs (3.13) if with probability 1, $x(n)$ satisfies (3.13) for all $n \in \mathbb{N}$ and $x(n)$ is $\mathcal{F}_n-$ measurable.

Proposition 3.3.3. *Let* $E, A \in \mathbb{C}^{n\times n}$ *satisfy* (3.14) *and* (1.20)*. Then the consistency condition of the perturbation for solvability of* (3.13) *is*

$$(I - E^D E)f = 0. \tag{3.18}$$

Moreover, the solution of equation (3.15b) *has only the form*

$$x_2(n) = 0, \text{ a.s. } \forall n \in \mathbb{N}. \tag{3.19}$$

Definition 3.3.4. The SSDE (3.13) is called tractable with index-ν (or for short, of index-ν) if

i) $\mathrm{Ind}(E, A) = \nu$,

ii) $(I - E^D E)f = 0$.

Next, we consider the dependence on the consistent initial condition of solution of equation (3.13). Assume that $\mathbb{R}^n$ is endowed with the Euclidean norm and there exists $b > 0$ such that

$$\|f(n, x) - f(n, 0)\| \leqslant b\|x\|, \text{ for all } x \in \mathbb{R}^n.$$

Hence, Proposition 3.3.3, we have $x_2(n) = 0$ implies $x(n) = x_1(n)$.

Let

$$\alpha := \sup_{0\leqslant n\leqslant N} \left\{\|E^D A\|^2 + 2b_n^2\|E^D\|^2\right\},$$

$$\beta := \sup_{0\leqslant n\leqslant N} \|E^D f(n, x(0))\|.$$

Theorem 3.3.5. *Let $x(n)$ be the unique solution of equation (3.13) and $\mathbb{E}\|x(0)\|^2 < \infty, n \in \mathbb{N}$. Then the following inequalities hold for all $0 \leqslant n \leqslant N$*

$$\mathbb{E}\|x(n)\|^2 \leqslant \alpha^n \mathbb{E}\|x(0)\|^2 + \frac{2\beta^2(\alpha^n - 1)}{\alpha - 1}. \tag{3.20}$$

3.3.2. Stability for stochastic singular difference equations of index-ν

In this subsection, we study stability of the SSDEs of index-ν. It is well known that the method of Lyapunov functions is very useful to investigate stability of dynamic systems. Let us consider the equations

$$\begin{cases} Ex(n+1) = Ax(n) + f(n, x(n))w_{n+1}, \\ x(0) = x_0, \end{cases} \tag{3.21}$$

where $E, A \in \mathbb{K}^{n \times n}$ are constant matrices.

Definition 3.3.6. The trivial solution of equation (3.21) is called:

- Mean square stable if for any $\varepsilon > 0$ and there exists a $\delta > 0$ such that $\mathbb{E}\|x(n)\|^2 < \varepsilon, \forall n \in \mathbb{N}$, if $\mathbb{E}\|EE^D x(0)\|^2 < \delta$.
- Asymptotically mean square stable if it is mean square stable and with $\mathbb{E}\|EE^D x(0)\|^2 < \infty$ the solution $x(n)$ of (3.21) satisfies $\lim_{n\to\infty} \mathbb{E}\|x(n)\|^2 = 0$.

If the trivial solution of equation (3.21) is mean square stable (resp. asymptotically mean square stable) then we say equation (3.21) is mean square stable (resp. asymptotically mean square stable).

Theorem 3.3.7. *Assume that there exist $a, b, c_2 > 0$ such that $||f(n,x)|| \leqslant b||x||$ and the matrix equation*

$$(EE^D A)^T HEE^D A - (EE^D E)^T HEE^D E = -a^2 (EE^D)^T EE^D \tag{3.22}$$

has a nonnegative definite solution H satisfying

$$b^2\|H\| - a^2 \leqslant -c_2 < 0. \tag{3.23}$$

Then the trivial solution of (2.9) is asymptotically mean square stable.

Conclusion

This thesis deals with two main problems.The following results have achieved:

1. Using the Drazin inverse approach to decouple the system into the differential and algebraic subsystems. The consistent condition of the perturbation f is also given.

2. Formulas of the stability radii for stochastic differential-algebraic equations with respect to stochastic perturbations are derived.

3. We have established the explicit expression of the solution, the variation of constants formula, and the continuous dependence on the initial condition.

4. Given the necessary conditions for the mean square stability of linear stochastic implicit difference equations by using the method of solution evaluation.

5. Given the concept of the mean square stable and asymptotically mean square stable, given theorems of necessary and sufficient conditions of stochastic singular difference equations, also investigated by using the method of Lyapunov functions.

Here are some of our future research directions:

1. Give conditions for the L^2-stable and exponentially L^2-stable as well as stable radii of time-varying stochastic differential-algebraic equations of index-1 and index-ν.

2. Studying the stability and other control properties in stochastic differential-algebraic equations.

3. Provides formulas to calculate the stable radii for stochastic difference implicit equations.

THE AUTHER'S PUBLICATIONS RELATED TO THE THESIS

1. Do Duc Thuan, Nguyen Hong Son and Cao Thanh Tinh (2021), "Stability radii of differential-algebraic equations with respect to stochastic perturbations, *Systems & Control Letters*, 147, pp. 1-9, https://doi.org/10.1016/ j.sysconle.2020. 104834 (SCI, Q1).

2. Do Duc Thuan, Nguyen Hong Son (2020), "Stochastic implicit difference equations of index-1", *J. Differ. Equations Appl.*, 26(11-12), pp. 1428-1449. (SCIE, Q2).

3. Nguyen Hong Son, Ninh Thi Thu (2020), "A comparison theorem for stability of linear stochastic implicit difference equations of index-1", *VNU Journal of Sicence Mathematics-Physics*, 36(3), pp. 24-31.

4. Do Duc Thuan, Nguyen Hong Son (2020), "Stochastic singular difference equations of index-ν", Preprint.