INFORMATION ON MASTER'THESIS

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7. Official thesis title
   "Some of nonlinear inequalities with discrete time"
8. Major: Primary Mathematics Method
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11. Summary of the finding of the thesis
    My research is about
    "Some of nonlinear inequalities with discrete time"
    It introduces some of basic concept, theorems about difference inequalities in one variable and in several in dependent variables.It has two chapters.
    Chapter 1. Difference Inequalities.
    It is well recognized that the inequalities furnish a very general comparison principle in studying many qualitative as well as quantitative properties of solutions of related equations. The celebrated Gronwall's inequality is a one of the examples for a monotone operator κ in which the exact solution of $w = p + \kappa w$ provides an upper bound on all solutions of the inequality $u \le p + \kappa u$.We begin this chapter with Gronwall type inequalities, next nonlinear inequalities and inequalities involving differences and finite systems of inequalities, finally we shall consider Opial and Wirtinger type inequalities.
    a. Gronwall inequalities
    Theorem : Let for all $k \in \mathbb{N}(a)$ the following inequality be satisfied

$$u(k) \le p(k) + q(k) \sum_{l=a}^{k-1} f(l)u(l). \tag{1}$$

Then, for all $k \in \mathbb{N}(a)$

$$u(k) \le p(k) + q(k) \sum_{l=a}^{k-1} p(l)f(l) \prod_{\tau=l+1}^{k-1} (1 + q(\tau)f(\tau)). \tag{2}$$

b. Nonlinear Inequalities
Theorem : Let for all $k \in \mathbb{N}(a)$ the following inequality be satisfied

$$u(k) \leq p(k) = \left[q + \sum_{i=1}^{r} H_i(k,u) \right],$$

where

$$H_i(k,u) = \sum_{l_1=a}^{k-1} f_{i1}(l_1) u^{\alpha_{i1}}(l_1) \cdots \sum_{l_i=a}^{l_{i-1}-1} f_{ii}(l_i) u^{\alpha_{ii}}(l_i)$$

and $a_{ij}, 1 \leq j \leq i, 1 \leq i \leq r$ are nonnegative constants and the constant $q > 0$. Then, for all $k \in \mathbb{N}(a)$

$$u(k) \leq qp(k) \prod_{l=a}^{k-1} (1 + \Delta Q(l)) \quad \text{nếu } \alpha = 1 \tag{3}$$

$$u(k) \leq p(k) \left[q^{1-\alpha} + (1-\alpha) Q(k) \right]^{1/1-\alpha} \quad \text{nếu } \alpha \neq 1 \tag{4}$$

where

$$Q(k) = \sum_{i=1}^{r} H_i(k,p) q^{\alpha_i - \alpha}$$

and when $\alpha > 1$, we assume that $q^{1-\alpha} + (1-\alpha)Q(k) > 0$ for all $k \in \mathbb{N}(a)$.

c. Inequalities involving differences

Theorem : Let fori $k \in \mathbb{N}(a)$ the following inequality be satisfied

$$\Delta^n u(k) \leq p(k) + q(k) \sum_{i=0}^{n} \sum_{l=a}^{k-1} q_i(l) \Delta^i u(l). \tag{5}$$

Then, for all $k \in \mathbb{N}(a)$

$$\Delta^n u(k) \leq p(k) + q(k) \sum_{l=a}^{k-1} \phi_1(l) \prod_{\tau=l+1}^{k-1} (1 + \phi_2(\tau)), \tag{6}$$

where

$$\phi_1(k) = p(k)q_n(k) + \sum_{i=0}^{n-1} \sum_{j=0}^{i} \Delta^i u(a) q_i(k) \frac{(k-a)^{(i-j)}}{(i-j)!} + \sum_{i=0}^{n-1} q_{n-i-1}(k) \sum_{l=a}^{k-i-1} \frac{(k-l-1)^{(i)}}{i!} p(l) \tag{7}$$

and

$$\phi_2(k) = q(k)q_n(k) + \sum_{i=0}^{n-1} q_{n-i-1}(k) \sum_{l=a}^{k-i-1} \frac{(k-l-1)^{(i)}}{i!} q(l). \tag{8}$$

d. Finite systems of inequalities

Let the subscript i range over the integers $1, \cdots, n$ and r be some fixed positive integer such that $1 < r < n$. The subscripts p and q range over the integers $1, \cdots, r$ and $r+1, \cdots, n$ respectively.

Definition 1. The function $f(k, \mathbf{u})$ is said to possess mixed monotone property if

- $f_p(k, \mathbf{u})$ is nondecreasing in $u_1, \cdots, u_r$ and nonincreasing in $u_{r+1}, \cdots, u_n$ for all fixed $k \in \mathbb{N}(a)$.
- $f_q(k, \mathbf{u})$ is nonindecreasing in $u_1, \cdots, u_r$ and nondecreasing $u_{r+1}, \cdots, u_n$.

In particular $\mathbf{f}(k,\mathbf{u})$ is said to possess nondecreasing property if $f_i(k,\mathbf{u})$ is nondecreasing in $u_1,\cdots,u_n$ for all fixed $k\in\mathbb{N}(a)$.
Definition 2. The function $\mathbf{v}(k)$ defined on $\mathbb{N}(a)$ is said to be a r under and $(n-r)$ over function with respect to the system $\mathbf{u}(k+1)=\mathbf{f}(k,\mathbf{u}(k))$ if $v_p(k+1)\geq f_p(k,\mathbf{v}(k))$ and $v_q(k+1)\leq f_q(k,\mathbf{v}(k))$ for all $k\in\mathbb{N}(a)$. If $\mathbf{v}(k)$ satisfies the reverse inequalities, then it is said to be r over and $(n-r)$ under function.
Theorem: Let the function $\mathbf{f}(k,\mathbf{u})$ possess mixed monotone property. Further, let there exist two functions $\mathbf{v}(k)$ and $\mathbf{w}(k)$ defined on $\mathbb{N}(a)$ such that

$$\begin{aligned} v_p(k+1)\leq f_p(k,\mathbf{v}(k)), &\quad v_q(k+1)\geq f_q(k,\mathbf{v}(k))\\ w_p(k+1)\geq f_p(k,\mathbf{w}(k)), &\quad w_q(k+1)\leq f_q(k,\mathbf{w}(k))\\ v_p(a)\leq w_q(a), &\quad v_q(a)\geq w_q(a). \end{aligned}\tag{9}$$

Then, for all $k\in\mathbb{N}(a)$

$$v_p(k)\leq w_p(k),\quad v_q(k)\geq w_q(k).\tag{10}$$

e. Opial type inequalities *Theorem* : Let $u(k)$ be nondecresing for all $k\in\mathbb{N}(a)$ and $u(a)=0$. Then,

(a) If $p>0,q>0,p+q\geq1$ or $p<0,q<0$

$$\sum_{l=a}^{k-1}(\Delta u(l))^q u^p(l+1)\leq H(k-a)\sum_{l=a}^{k-1}(\Delta u(l))^{p+q},\tag{11}$$

where $H(0)=q(p+q)^{-1}$ and for $k\in\mathbb{N}(a+1)$

$$H(k-a)=max\left\{H(k-a-1)+\frac{p(k-a)^{p-1}}{(p+q)},\frac{q(k-a+1)^p}{(p+q)}\right\}.$$

(b) If $p>0,q<0,p+q\leq1,p+q\neq0$ or $p<0,q>0,p+q\geq1$

$$\sum_{l=a}^{k-1}(\Delta u(l))^q u^p(l+1)\geq h(k-a)\sum_{l=a}^{k-1}(\Delta u(l))^{p+q},\tag{12}$$

where $h(0)=q(p+q)^{-1}$ and for $k\in\mathbb{N}(a)$

$$h(k-a)=min\left\{h(k-a-1)+\frac{p(k-a)^{p-1}}{(p+q)},\frac{q(k-a+1)^p}{(p+q)}\right\}.$$

(c) If $p\geq1,q\geq1$ then (9) holds with $H(k-a)$ replaced by $q(k-a+1)^p(p+q)^{-1}$.

(d) If $p\leq0,q<0$ then (9) holds with $H(k-a)$ replaced by $J(k-a)$, where $J(0)=q(p+q)^{-1}$ and for $k\in\mathbb{N}(a+1)$

$$J(k-a)=1+p(p+q)^{-1}\sum_{l=a+2}^{k}(l-a)^{p-1}.$$

(e) If $p\geq0,p+q<0$ then (10) holds with $h(k-a)$ replaced by $J(k-a)$.

f. Wirtinger type inequalities
Theorem : Let $\theta_i\in(0,l),p_i>0,i=1,\cdots,n,P_n=\sum_{i=1}^n p_i$, và $\sigma=(1/P_n)\sum_{i=1}^n p_i\theta_i$. Let $f(\theta)$ be a positive $C^{(2)}(0,l)$ function such that $f'(\theta)f''(\theta)\neq0$ on $(0,l)$ and

$$[f'(\theta)]^2-f(\theta)f''(\theta)=\mu,\ \ 0<\theta<1\tag{13}$$

where μ is a constant

(a) If $f''(\theta) < 0$ on $(0, l)$, then

$$\left(\sum_{i=1}^{n} p_i f(\theta_i)\right)^2 - c_n \sum_{i=1}^{n} p_i f(\theta_i) f'(\theta_i) \geq \left(P_n f(\sigma) - \sum_{i=1}^{n} p_i f(\theta_i)\right)^2, \qquad (14)$$

(b) If $f''(\theta) > 0$ on $(0, l)$, then

$$\left(\sum_{i=1}^{n} p_i f(\theta_i)\right)^2 - c_n \sum_{i=1}^{n} p_i f(\theta_i) f'(\theta_i) \leq \left(P_n f(\sigma) - \sum_{i=1}^{n} p_i f(\theta_i)\right)^2, \qquad (15)$$

where $c_n = P_n f(\sigma)/f'(\sigma)$.

In (12) và (13) equality holds if and only if $\theta_i = \cdots = \theta_n = \sigma$.

Chapter 2 : Difference inequalities in several independent variables
Inequalities developed in chapter 1 have nature extensions for functions of m independent variables. These inequalities are used as a fundamental tool in the study of related partial difference equation. We begin this chapter with some concept about discrete Riemann's function. This function is used to study linear Gronwall and Wendroff inequalities. Next is nonlinear inequalities and inequalities involving higher order differences in two independent variables. Later, we move to muntidimensional linear as well as nonlinear inequalities. Finally, we shall develop Opial's and Wirtinger's type inequalities in two independent variables.

a. Discrete Riemann's Function.

Once again, let $\mathbb{N} = \{0, 1, \cdots\}$ be the set of nature numbers including zero, and the product $\mathbb{N} \times \cdots \times \mathbb{N}$ (m times) be denotes by $\mathbb{N}^m$. A point $(x_1, \cdots, x_m)$ in $\mathbb{N}^m$ is denoted by x, whereas $\overline{x}_i$ represents $(x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_m)$, and $(\overline{x}_i, \bullet)$ stands for $(x_1, \cdot\cdot\cdot, x_{i-1}, \bullet, x_{i+1}, \cdot\cdot\cdot, x_m)$,also $s, x \in \mathbb{N}^m$, $0 \leq s \leq x$ represents $0 \leq s_i \leq x_i$, $1 \leq i \leq m$. For a given function $u(x)$ on $\mathbb{N}^m$, the first order difference with respect to the variabe x_i is defined as $\Delta_{x_i} u(x) = u(\overline{x}_i, x_i + 1) - u(k)$, and the second order difference with respect to the variables x_i and x_j is defined as $\Delta_{x_i}\Delta_{x_j} u(x) = \Delta_{x_i} u(\overline{x}_j, x_j + 1) - \Delta_{x_i} u(x) = u(x_1, \cdot\cdot\cdot, x_{i-1}, x_i + 1, x_{i+1}, \cdot\cdot\cdot, x_{j-1}, x_j + 1, x_{j+1}, \cdot\cdot\cdot, x_m) - u(\overline{x}_i, x_i + 1) - u(\overline{x}_j, x_j + 1) + u(x)$. The higher order differences are defined analogously. The $\mathbf{S}_{l=s}^{x-1} u(l)$ represents the m fold sum $\sum_{l_1=s_1}^{x_1-1} \cdots \sum_{l_m=s_m}^{x_m-1} u(l_1, \cdot\cdot\cdot, l_m)$, and $\Delta_x^m u(x)$ denotes $\Delta_{x_1} \cdots \Delta_{x_m} u(x_1, \cdot\cdot\cdot, x_m)$. The empty sums and products are taken to be 0 and 1, respectively.

b. Linear Inequalities.

In what follows we shall assume that the functions which appear in the inequalities are real valued, nonnegative and defined on $\mathbb{N}^m$.

Theorem : Let for all $x \in \mathbb{N}^m$ the following inequality be satisfied

$$u(x) \leq p(x) + q(x)\mathbf{S}_{s=0}^{x-1} f(s)u(s). \qquad (16)$$

Then, for all $x \in \mathbb{N}^m$

$$u(s) \leq p(x) + q(x)\mathbf{S}_{s=0}^{x-1} f(s)p(s)V(s+1; x), \qquad (17)$$

where $V(s; x)$ is the solution of

$$(-1)^m \Delta_s^m V(s; x) = f(s)q(s)V(s+1; x), \quad s \leq x - 1$$
$$V(\overline{s}_i, x_i; x) = 1, \quad 1 \leq i \leq m.$$

c. Wendroff Type Inequalities.
Let $W(s;x)$ be any function defined for all $s \le x-1$, $(s;x) \in \mathbb{N}^m \times \mathbb{N}^m$ and

$$\begin{aligned} &(-1)^m \Delta_s^m W(s;x) \ge f(s)q(s)W(s+1;x), \quad s \le x-1 \\ &W(\overline{s}_i, x_i; x) = 1, \quad 1 \le i \le m. \end{aligned} \tag{18}$$

Then, from Lemma 2.1.4 it follows that in (2.1.4), $V(s+1;x)$ can be replaced by $W(s+1;x)$. However, finding a suitable $W(s;x)$ in advance which satisfies (2.23) seems to be quite difficult. Therefore, for the function $V(s;x)$ we shall provide an upper estimate which is quite adequate in practical applications.
Lemma: Let $V(s;x)$ be as in Theorem 2.2.1. Then, for all $s \le x-1, (s;x) \in \mathbb{N}^m \times \mathbb{N}^m$

$$V(s;x) \le \prod_{l_1=s_1}^{x_1-1} \left(1 + \mathbf{S}_{\overline{l}_1=\overline{s}_1}^{\overline{x}_1-1} f(l)q(l)\right). \tag{19}$$

d. Nonlinear Inequalities.
Theorem: Let for all $x \in \mathbb{N}^m$ the following inequality be satisfied

$$u(x) \le p(x)\left[q(x) + \sum_{i=1}^{r} H_i(x,u)\right], \tag{20}$$

where

$$H_i(x,u) = \mathbf{S}_{x^1=0}^{x-1} f_{i1}(x^1) u^{\alpha_1 i}(x^1) \cdots \mathbf{S}_{x^i=0}^{x^{i-1}-1} f_{ii}(x^i) u^{\alpha_i i}(x^i) \tag{21}$$

and $\alpha_{ij}, 1 \le j \le i, 1 \le i \le r$ are nonnegative constants and the constant $q > 0$. We shall denote $\alpha_i = \sum_{j=1}^{i} \alpha_{ij}$ and $\alpha = \max\limits_{1 \le j \le r} \alpha_i$. Then, for all $x \in \mathbb{N}^m$

$$u(x) \le qp(x) \min_{1 \le i \le m} \left\{ \prod_{l_i=0}^{x_i-1} (1 + \Delta_{l_i} Q(\overline{x}_i, l_i)) \right\}, \quad \text{if } \alpha = 1 \tag{22}$$

and

$$u(x) \le p(x)[q^{1-\alpha} + (1-\alpha)Q(x)]^{1/(1-\alpha)}, \quad \text{if } \alpha \ne 1 \tag{23}$$

where $Q(x) = \sum\limits_{i=1}^{r} H_i(x,p) q^{\alpha_i - \alpha}$ and when $\alpha > 1$, we assume that $q^{1-\alpha} + (1-\alpha)Q(x) > 0$ for all $x \in \mathbb{N}^m$.
e. Inequalities Involving Partial Differences.
Theorem: Let for all $k, l \in \mathbb{N} \times \mathbb{N}$ the following inequality be satisfied

$$\Delta_k^{r_1} \Delta_l^{r_2} u(k,l) \le p(k,l) + q(k,l) \sum_{i=0}^{r_1} \sum_{j=0}^{r_2} \sum_{\tau=0}^{k-1} \sum_{\eta=0}^{l-1} h_{ij}(\tau,\eta) \Delta_\tau^i \Delta_\eta^j u(\tau,\eta). \tag{24}$$

Then, for all $(k,l) \in \mathbb{N} \times \mathbb{N}$

$$\Delta_k^{r_1} \Delta_l^{r_2} u(k,l) \le p(k,l) + q(k,l) \sum_{\tau=0}^{k-1} \sum_{\eta=0}^{l-1} A_1(\tau,\eta) V(\tau+1, \eta+1; k, l),. \tag{25}$$

where

$$A_1(k,l) = h_{r_1r_2}(k,l)p(k,l) + \sum_{j=0}^{r_2-1} h_{r_1j}(k,l)\times$$

$$\left[\sum_{\beta=j}^{r_2-1} \frac{(l)^{(\beta-j)}}{(\beta-j)!}\Delta_k^{r_1}\Delta_l^{\beta}u(k,0) + \frac{1}{(r_2-j-1)!}\sum_{\eta=0}^{l-r_2+j}(l-\eta-1)^{(r_2-j-1)}p(k,\eta)\right]$$

$$+\sum_{i=0}^{r_1-1} h_{ir_2}(k,l)\times\left[\sum_{\alpha=i}^{r_1-1}\frac{(k)^{(\alpha-i)}}{(\alpha-i)!}\Delta_k^{\alpha}\Delta_l^{r_2}u(0,l) + \frac{1}{(r_1-i-1)!}\times\sum_{\tau=0}^{k-r_1+i}(k-\tau-1)^{(r_1-i-1)}p(\tau,l)\right]$$

$$+\sum_{i=0}^{r_1-1}\sum_{j=0}^{r_2-1} h_{ij}(k,l)\times$$

$$\left[\phi_{ij}(k,l) + \frac{1}{(r_1-i-1)!(r_2-j-1)!}\sum_{\tau=0}^{k-r_1+i}\sum_{\eta=0}^{k-r_2+j}(k-\tau-1)^{(r_1-i-1)}(l-\eta-1)^{(r_2-j-1)}p(\tau,\eta)\right] \tag{26}$$

and $V(\tau,\eta;k,l), \tau \le k-1, \eta \le l-1$ is the solution of

$$\Delta_\tau\Delta_\eta V(\tau,\eta;k,l) = B_1(\tau,\eta)V(\tau+1,\eta+1;k,l) \tag{27}$$

$$V(k,\eta;k,l) = V(\tau,l;k,l) = 1, \tag{28}$$

where

$$\begin{aligned} B_1(k,l) &= h_{r_1r_2}(k,l)q(k,l) \\ &+ \sum_{j=0}^{r_2-1} h_{r_1j}(k,l)\frac{1}{(r_2-j-1)!}\sum_{\eta=0}^{l-r_2+j}(l-\eta-1)^{(r_2-j-1)}q(k,\eta) \\ &+ \sum_{i=0}^{r_1-1} h_{ir_2}(k,l)\frac{1}{(r_1-i-1)!}\sum_{\tau=0}^{k-r_1+i}(k-\tau-1)^{(r_1-i-1)}q(\tau,l) \\ &+ \sum_{i=0}^{r_1-1}\sum_{j=0}^{r_2-1} h_{ij}(k,l)\frac{1}{(r_1-i-1)!(r_2-j-1)!}\times \\ &\sum_{\tau=0}^{k-r_1+i}\sum_{\eta=0}^{k-r_2+j}(k-\tau-1)^{(r_1-i+1)}(l-\eta-1)^{(r_2-j-1)}q(\tau,\eta). \end{aligned} \tag{29}$$

f. Multidimensional Linear Inequalities.

Theorem: Let the $n \times n$ matrices $\mathbf{G}(x)$ and $\mathbf{H}(x)$ be defined and nonengative on $\mathbb{N}^m$, and the n vector functions $\mathbf{p}(x)$ and $\mathbf{u}(x)$ be defined on $\mathbb{N}^m$. Further, let for all $x \in \mathbb{N}^m$ the following inequality be satisfied

$$\mathbf{u}(x) \le \mathbf{p}(x) + \mathbf{G}(x)\mathbf{S}_{s=0}^{x-1}\mathbf{H}(s)\mathbf{u}(s). \tag{30}$$

Then, for all $x \in \mathbb{N}^m$

$$\mathbf{u}(x) \le \mathbf{p}(x) + \mathbf{G}(x)\mathbf{S}_{s=0}^{x-1}\mathbf{V}(s+1;x)\mathbf{H}(s)\mathbf{p}(s), \tag{31}$$

where $\mathbf{V}(s;x)$ satisfies

$$\mathbf{V}(s;x) = \mathbf{X} + \mathbf{S}_{l=s}^{x-1}\mathbf{V}(l+1;x)\mathbf{H}(l)\mathbf{G}(l). \tag{32}$$

g. Multidimensional Nonlinear Inequalities.
In this section we are concerned with comparing the solution $\mathbf{u}(x), x \in \mathbb{N}^m$ of the nonlinear difference equation

$$\Delta_x^m \mathbf{u}(x) = \mathbf{f}(x, \mathbf{u}(x)) \tag{33}$$

with solutions $\mathbf{v}(x)$ and $\mathbf{w}(x)$ of the corresponding nonlinear difference inequalities

$$\Delta_x^m \mathbf{v}(x) \leq \mathbf{f}(x, \mathbf{v}(x)) \tag{34}$$

and

$$\Delta_x^m \mathbf{w}(x) \geq \mathbf{f}(x, \mathbf{w}(x)), \tag{35}$$

respectively.
In what follows $(i)x$ denotes a point $(x_1, \cdots, x_m)$ in which i variables are zero. There are $\begin{pmatrix} m \\ i \end{pmatrix}$ total such possibilities. Thus, if at the m hyperplanes $x = (1)x$ the function $\mathbf{u}(x)$ is known, then a recursive argument can be used to ensure the existence and uniqueness of the solutions of (31). This is apparent from the summation representation

$$\mathbf{u}(x) = \sum_{i=1}^{m} (-1)^{i+1} \sum_{i} \mathbf{u}((i)x) + \mathbf{S}_{s=0}^{x-1} \mathbf{f}(s, \mathbf{u}(s)), \tag{36}$$

where $\sum_i$ represents the summation over all the possibilities $(i)x$. From these notations it is also clear that the solutions $\mathbf{v}(x)$ and $\mathbf{w}(x)$ of the inequalities (32) and (33) have the summation representation

$$\mathbf{v}(x) \leq \sum_{i=1}^{m} (-1)^{i+1} \sum_{i} \mathbf{v}((i)x) + \mathbf{S}_{s=0}^{x-1} \mathbf{f}(s, \mathbf{v}(s)) \tag{37}$$

and

$$\mathbf{w}(x) \geq \sum_{i=1}^{m} (-1)^{i+1} \sum_{i} \mathbf{w}((i)x) + \mathbf{S}_{s=0}^{x-1} \mathbf{f}(s, \mathbf{w}(s)). \tag{38}$$

h. Opial and Wirtinger Type Inequalities in Two Variables.
Theorem: Let r_1 and r_2 be fixed positive integers and $u(k,l)$ be a function defined on $\mathbb{N} \times \mathbb{N}$ such that $u(k,l) = 0$ for all $k, l \in \mathbb{N}, 0 \leq k \leq r_1 - 1, 0 \leq l \leq r_2 - 1$. Then, for $0 \leq i \leq r_1 - 1, 0 \leq j \leq r_2 - 1$ và $(k,l) \in \mathbb{N} \times \mathbb{N}$

$$\begin{aligned} &\sum_{\tau=1}^{k-r_1+i} \sum_{\eta=1}^{l-r_2+j} |\Delta_\tau^i \Delta_\eta^j u(\tau + r_1 - i - 1, \eta + r_2 - j - 1)||\Delta_\tau^{r_1} \Delta_\tau^{r_2} u(\tau, \eta)| \\ &\leq \frac{1}{2\sqrt{2}(r_1 - i)(r_2 - j)!} \left(\frac{r_1 - i}{2r_1 - 2i - 1} \right)^{1/2} \left(\frac{r_2 - j}{2r_2 - 2j - 1} \right)^{1/2} \\ &\times (k)^{(r_1 - i)} (l)^{(r_2 - j)} \sum_{\tau=0}^{k-r_1+i} \sum_{\eta=0}^{l-r_2+j} |\Delta_\tau^{r_1} \Delta_\eta^{r_2} u(\tau, \eta)|^2. \end{aligned} \tag{39}$$

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