VIETNAM NATIONAL UNIVERSITY HANOI
UNIVERSITY OF SCIENCE

SOCIALIST REPUBLIC OF VIETNAM
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INFORMATION ON DOCTORAL THESIS

1. Full name: Quan Thai Ha 2. Sex: Male

3. Date of birth: 03/7/1980 4. Place of birth: Phu Tho

5. Admission decision number: 2741/QĐ-ĐHKHTN dated 19/7/2019 by University of Science, Vietnam National University Hanoi

6. Changes in academic process:

7. Official thesis title: **Theory and applications of convolution associated with the Hartley integral transform**

8. Major: Applied mathematics 9. Code: 9460112.01

10. Supervisors: Prof. DSc. Pham Ky Anh and Assoc. Prof. Dr. Nguyen Minh Tuan

11. Summary of the new findings of the thesis:

The thesis investigates the Hartley transforms (HT) and their associated convolutions, aiming to contribute to the construction of theoretical foundations as well as to exploit their application potentials. The thesis delves into the application of HT in solving integral equations, particularly Fredholm integral equations (FIE) with a Toeplitz-plus-Hankel (TPH) kernel. Furthermore, the thesis studies methods for obtaining exact and approximate solutions to FIEs involving linear TPH parts, for both linear and nonlinear cases.

Based on functional analysis and Fourier analysis, the thesis systematically studies and develops the theoretical foundations for HT, aiming to contribute to a more comprehensive and rigorous theoretical framework for HT.

- Operator theory: Investigate isometric properties (energy preservation). Discover and analyze basis functions and eigenfunctions.
- Relationship between the domain of origin and the transform domain: Determine the relationship between differential and integral operations in the domain of origin and algebraic and convolution operations in the transform domain. Establish the connection between the smoothness of functions in the domain of origin and the decay rate of functions in the transform domain.

- The thesis establishes convolution theorems, norm inequalities, and uncertainty principles.

Based on the theory of monotone operators, multi-valued analysis, and convex analysis, the thesis investigates methods used in optimization problems, aiming to apply and develop methods and algorithms for solving both linear and nonlinear integral equations.

- Investigation of numerical methods for solving integral equations: Study iterative methods, projection methods, splitting methods, and regularization methods.

Based on the applications of the FT and Fourier convolution, the thesis investigates the applicability of the HT as well as their associated convolutions. Furthermore, it examines the advantages that HT offers compared to FT.

Driven by the need to optimize computational performance in real signal processing, the HT - a unique tool operating entirely in the real domain - has attracted significant interest from many scientists. The Hartley transform is closely related to the FT; while FT utilizes a complex exponential integral kernel, HT employs a cas (cosine-and-sine) integral kernel. Although FT possesses a vast theoretical framework and plays an indispensable role as a foundational mathematical tool across numerous scientific and engineering fields, the self-inverse property and the complex-free nature of HT still offer distinct advantages that warrant thorough investigation.

The thesis investigates the HT, systematizing and developing the operator properties and operator characteristics of HT on the Schwartz space $S(\mathrm{R}^n)$, the Banach space $\mathrm{L}^1(\mathrm{R}^n)$, and the Hilbert space $\mathrm{L}^2(\mathrm{R}^n)$, with a particular focus on constructing convolutions associated with HT. On this basis, the thesis explores the application potential of these convolutions, especially in solving Fredholm integral equations (FIE). Specifically, the thesis focuses on studying methods to solve Fredholm integral equations of the second kind (FIE-II) - with a special case being FIE-II with a TPH kernel, and Fredholm integral equations of the first kind (FIE-I) involving linear TPH parts, for both linear and nonlinear cases.

The main contributions of the thesis include:

1. Systematizing and developing the operator properties and operator characteristics of HT on Schwartz spaces $S(\mathrm{R}^n)$, the Banach spaces $\mathrm{L}^1(\mathrm{R}^n)$ and the Hilbert spaces $\mathrm{L}^2(\mathrm{R}^n)$.

2. Constructing convolutions associated with HT and finite HT, while proving their crucial mathematical properties such as norm inequalities, factorization identities, and Young's inequalities; on this basis, the thesis constructs the Wiener algebra associated with HT on the underlying space $L^1(R^n)$.

3. Investigating several classes of FIE-II and deriving their exact solutions in closed form by utilizing the convolutions associated with HT. Specifically, the study establishes exact closed-form solutions for classes of FIE-II featuring TPH kernels, including both standard and shifted TPH kernels, through the application of convolutions linked to the finite HT. Furthermore, the thesis explores FIE-II on infinite domains, providing closed-form solutions for HT-associated convolution equations, shifted TPH kernel equations, and convolution equations involving Hermite functions.

4. Proposing the application of certain regularization methods to obtain approximate solutions for FIE-I involving linear TPH parts, for both linear and nonlinear cases. Specifically, The thesis proposes the application of several iterative regularization methods, integrated with convolutions associated with the finite HT to obtain approximate solutions for classes of FIE-I involving linear TPH parts. Both linear and nonlinear cases are addressed through techniques, including parallel iterative regularization, the forward-backward splitting method (FBSM), and the forward-backward-forward splitting method (FBFSM).

   The results of the thesis contribute to enriching functional analysis and operator theory, as well as opening up new application potentials and optimizing computational performance for problems in science and engineering, particularly in the field of real data processing.

12. Futher research directions:

Theoretical Mathematical Research:

- Investigate the mathematical properties of FTTs on abstract spaces, such as $L^p(R^n)$ spaces; test function spaces, including the space of infinitely differentiable functions with compact support $D(R^n)$ and the Schwartz space $S(R^n)$; distribution spaces, including the space of generalized functions $D'(R^n)$ and the space of tempered distributions $S'(R^n)$; and Schauder spaces. Although numerous studies have been conducted on integral transforms such as LCT and

OLCT on these function spaces, their degree of completeness remains significantly limited compared to FT.

- Investigate generalized forms of FTTs to enhance the flexibility of applying FTTs in practice.

Algorithm Development Research:

- Investigate and develop algorithms for efficient computation with HT in particular and FTTs in general, aiming to optimize computational performance across diverse hardware architectures (CPUs, GPUs, FPGAs) and for varying data scales.
- Investigate and develop algorithms in multi-dimensional spaces. Research and develop sparse algorithms for the efficient processing of sparse signals, thereby reducing computational time when signals contain numerous zero values.

Applied Research:

- Investigate and develop applications of FTTs in fields such as applied mathematics, signal processing, image and video processing, optics, cryptography, and machine learning.
- Investigate methods for solving integral equations with a TPH kernel, utilizing operator theory methods, iterative methods, and regularization techniques.
- Continue researching the application of HT to obtain approximate solutions for FIE-I with a TPH kernel, employing an approach based on bilevel optimization theory.

13. Thesis-related publications:

[CT1] N.M. Tuan, B.T. Giang, and Q.T. Ha (2025), "New Convolutions for the Hartley Integral Transform Imbedded in the Banach Algebras and Convolution-type Integral Equations", Demonstratio Mathematica vol. 58 (1), pp. 20240087. DOI: https://doi.org/10.1515/dema-2024-0087.

[CT2] V.T. Dung, and Q.T. Ha (2021), "Approximate Solution for Integral Equations InvolvingLinear Toeplitz plus Hankel Parts", Computational and Applied Mathematics vol. 40 (172). DOI: https://doi.org/10.1007/s40314-021-01558-8.

Hanoi, day ..... month.....year.....

On behalf of academic supervisors	**PhD. Student**
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