VIETNAM NATIONAL UNIVERSITY, HANOI

VNU UNIVERSITY OF SCIENCE

HA TUAN DUNG

SOME ASPECTS OF GEOMETRIC FLOWS IN RIEMANNIAN GEOMETRY

Speciality: Mathematical Analysis

Speciality code: 9460101.02

DISSERTATION

SUBMITTED IN PARTIAL FULFILLMENT OF THE REQUIREMENTS

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Assoc. Prof. TRAN THANH HUNG

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Confirmation

I hereby affirm that all scientific results presented in my dissertation were conducted under the dedicated guidance of Assoc. Prof. Nguyen Thac Dung and Assoc. Prof. Tran Thanh Hung. All findings included in this work are entirely original, truthful, and have not been previously published by any other individual. All contributions from co-authors' publications have been included with their full consent. I take full responsibility for the authenticity and integrity of the research outcomes detailed in my dissertation.

March 14, 2025
The author

Ha Tuan Dung

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Contents

Notation and Symbols

$\otimes$	tensor product
$(\cdot)^\top$	tangential projection
$(\cdot)^\perp$	normal projection
∂_i	i-th coordinate basis element or partial derivative with respect to x^i : $\frac{\partial}{\partial x^i}$
∂_t	partial derivative with respect to "time", t : $\frac{\partial}{\partial t}$
∇_i	covariant derivative as directional derivative of functions on the frame bundle
Hess or ∇^2	Hessian
Δ	Laplacian
Δ_f	weighted Laplacian (or f-Laplacian)
Δ_L	rough Laplacian (acting on tensors)
$\langle \cdot, \cdot \rangle$	Riemannian metric or inner product
$B_R(o)$ or $B(o, R)$	geodesic ball of radius R centered at o
$\mathcal{C}^k(M)$	space of all k-times differentiable functions with their derivatives of order k are continuous on M.
$\mathcal{C}^\infty(M)$	space of all smooth functions on M
$\mathcal{C}_c^\infty(M)$	space of all smooth functions with compact support on M
dim	dimension
dist	Riemannian distance
div	divergence
$d\mu$	Riemannian volume element
df	differential of the function f
exp	exponential function
g	Riemannian metric
$g(t)$ or $g(x, t)$	time-dependent metric
GRS	gradient Ricci soliton(s)

H	mean curvature
I	a "time" non-degenerate interval
$\mathrm{Iso}(M)$	isometry group of M
$\mathfrak{iso}(M)$	Lie algebra of the isometry group of M
$\mathcal{L}_X$	Lie derivative with respect to the vector field X
ln	natural logarithm
M^n	n-dimensional manifold
PDE	partial differential equation(s)
ϕ^*	pullback of the map ϕ
$\omega^\natural$	dual vector field to the 1-form ω
σ_n	volume of the unit Euclidean n-ball
$\mathbb{R}$	the set of real numbers
$\mathbb{R}^n$	n-dimensional Euclidean space
Rm	Riemann curvature operator
Ric	Ricci curvature
Ric_f^m	m-Bakry-Émery curvature
Ric_f	(∞)-Bakry-Émery curvature
S	scalar curvature
tr or trace	trace
TP	tangent bundle of P

Chapter 1

Introduction

The field of geometric flows is one of the most important areas of geometric analysis, forming at the nexus of differential equations and geometry. This field of study is characterized by the deformation of geometric objects such as metrics, mappings, and submanifolds by geometric attributes such as curvature and consists of partial differential equations (PDEs) of parabolic type. These flows have wide applications in many scientific fields. For example, in cell biology, they aid in understanding dynamic network rewiring during cellular differentiation and cancer (see [10]); in medical imaging, they used to conformal brain mapping and virtual colonoscopy (see [115]); in computer graphics, they help model vorticity lines for efficient smoke and dust animations in games and CGI effects (see [24, 72]); and in physics, they can model dynamic systems and space-time geometries (see [75, 93]). In pure mathematics, geometric flows have demonstrated their great potential by solving various problems related to differential geometry and topology (see [5, 69, 73, 78]). The field of geometric flows can be seen as a bridge between analysis and geometry. Moreover, thanks to this intersection, researchers can use tools and methods from the theory of PDEs, differential geometry, or both to study challenging problems in this field.

This field's starting point can come from Mullins's work in 1956. He proposed the curve shortening flow to model the motion of idealized grain boundaries in [75]. However, the field became widely known through Eells-Sampson's seminal paper [38] on the harmonic map heat flow in 1964. Specifically, in this paper, they established harmonic map heat flow and used it to prove the existence of harmonic maps into targets with nonpositive sectional curvature. From the perspective of Eells-Sampson's paper, we can roughly understand that a geometric flow deforms a geometric object over time via a differential equation, refining the object to make it more comprehensible or better suited to a specific purpose.

In the PDEs theory, investigating special solutions, such as radial or stable solutions, plays an important role in establishing qualitative and quantitative properties for the general solutions of the equation under consideration. These solutions are either expressible in closed form or, if not feasible, will be systematically classified. Solitons in geometric flows are a typical example of such special solutions. They remain invariant in time to a certain degree under a particular flow. A basic example of these solitons would be a family of round spheres in Euclidean space, which gradually shrink in size over time and eventually collapse to a single point. This behavior serves as a solution to the mean curvature flow, a type of geometric flow that evolves shapes by smoothing them out. On the other hand, as the geometric flow progresses, it can lead to intricate geometric changes, including the appearance of singularities, where quantities containing the norm of the curvature tensor approach to infinity, typically forming in finite time, due in part to the nonlinearity of geometric flow equations, as well as for geometric and topological reasons. Solitons of some geometric flows, such as Ricci flows and mean curvature flows, serve as prototypical singularity models. This is also one of the main motivations to promote further research by mathematicians in this topic and the field of geometric flows in general.

This dissertation investigates some aspects of geometric flows, with a particular focus on two main research directions as follows.

- The first aim is to study some geometric and topological properties of gradient Ricci solitons and translating solitons.

- The second aim is to explore the analytical aspects of some partial differential equations that originate from geometry within the context of some super geometric flows.

In the following three subsections of this chapter, we will provide an overview of the problems studied in the dissertation. The content of this chapter is essentially adapted from [26, 28, 46, 69] and the introductory sections of the papers that make up my dissertation [32, 33, 34, 35].

1.1 Gradient Ricci soltions and isometry groups

The Ricci flow equation is a geometric evolution equation that deforms the metric g of a Riemannian manifold over time by adjusting it in a way proportional to the Ricci curvature Ric:

$$\frac{\partial g}{\partial t} = -2\text{Ric}. \tag{1.1}$$

A Ricci flow (or a solution to the above equation) is a one-parameter family of metrics g, defined on a smooth manifold M and parameterized by t within a non-degenerate interval I, that satisfies the equation (1.1). The Ricci flow was introduced in 1982 by Hamilton [41] as part of his ambitious program to prove Poincaré's conjecture and Thurston's geometrization conjecture (see also [43]). Since then, it has been a primary object of study in the field of geometric flows and a groundbreaking tool for solving complex problems in pure mathematics such as Poincaré's conjecture, Thurston's geometrization conjecture, the Differentiable sphere theorem [15, 16] and a version of this theorem for the curvature of the second kind [20], or the generalized Smale conjecture [5, 7]. For an overview of recent advancements in the theory of Ricci flow, we refer the readers to the survey paper of R. Bamler [5] and the references therein.

In the paper [41], using the Ricci flow, Hamilton proved that if M is a compact 3-manifold that admits a Riemannian metric with strictly positive Ricci curvature, then M also admits a metric of constant positive curvature. As pointed out by him, this result strongly links to Poincaré's conjecture on compact, simply connected 3-manifolds and Smith's conjecture concerning the group of covering transformations [92]. If both conjectures hold, the result would naturally follow as a corollary. Furthermore, it is essential to realize that the Ricci flow equation is only weakly parabolic, often leading to finite-time singularities. Hamilton and many mathematicians have found that proving the Poincaré conjecture using Ricci flow requires overcoming the challenges posed by singularity models of this flow. This has prompted the study of singularity models to gain insight into the underlying topological and geometric features of Ricci flows. Probably the most important singularity model is the Ricci soliton, which is a self-similar solution to the Ricci flow equation (1.1) and arises as a finite-time singularity model. Recall that a Ricci soliton is a Riemannian manifold (M, g) that is equipped with a smooth vector field X satisfying the equation

$$\text{Ric} + \frac{1}{2}\mathcal{L}_X g = \lambda g, \tag{1.2}$$

where $\mathcal{L}$ is the Lie derivative with respect to X and $\lambda \in \mathbb{R}$. In particular, if $X = \nabla f$ where $f : M \to \mathbb{R}$ is a smooth function, then we say that a triple

(M, g, f) is a gradient Ricci soliton. In this case the equation (1.2) becomes

$$\text{Ric} + \text{Hess}\, f = \lambda g, \tag{1.3}$$

where Hess is the Hessian of metric g. Depending on the value of λ, a gradient Ricci soliton is called shrinking if $\lambda > 0$, steady if $\lambda = 0$, or expanding if $\lambda < 0$.

An Einstein manifold N is a Riemannian manifold whose Ricci curvature Ric of N is proportional to the metric g of N, that is $\text{Ric} = \lambda g$, where λ is a fundamental constant. Here λ is called the Einstein constant. These manifolds play a central role in differential geometry and theoretical physics, particularly in general relativity, where they model space-times with constant curvature. It is not hard to see that an Einstein manifold is a basic example of gradient Ricci soliton where the Hessian operator acting on the potential function f equals zero and λ becomes the Einstein constant. Another basic example is the Gaussian soliton $\left(\mathbb{R}^n, g_{\mathbb{R}^n}, \frac{\lambda|x|^2}{2}\right)$, followed by cylinders $\mathbb{S}^k \times \mathbb{R}^{n-k}$ with the product metric where the sphere has Ricci curvature λ. Furthermore, a combination of the two mentioned earlier, as the notation of Petersen and Wylie [81], is referred to as a *rank k rigid* gradient Ricci soliton. In particular, it is isometric to an appropriate quotient of $N^k \times \mathbb{R}^{n-k}$, with $f = \frac{|x|^2}{2}$ defined on the Euclidean factor [80]. Consequently, a soliton is called *non-trivial* (or *non-rigid*) if at least a factor in its de Rham decomposition is non-Einstein.

On the other hand, the study of isometric groups plays a pivotal role in classifying the geometric structure of smooth manifolds. Dantzig-Waerden's groundbreaking paper [30] nearly a century ago on the group of isometries of a connected, locally compact metric space can be seen as the starting point for a series of works on this subject. Myers and Steenrod in [76] showed that the isometry group $\text{Iso}(M)$ of a Riemannian manifold M is a Lie transformation group concerning the compact-open topology. Later, Kobayashi [53] determined the maximal dimension of $\text{Iso}(M)$ and showed that the Riemannian manifold M is of constant curvature [53], provided the dimension of $\text{Iso}(M)$ is maximal. While (non-gradient) Ricci solitons have been found in various Lie groups and homogeneous spaces [9, 58], Petersen and Wylie [82] proved that all homogeneous gradient Ricci solitons are rigid. Furthermore, they also demonstrated that if the Riemannian metric is reducible, the soliton structure is also reducible. Their result is based on the existence of splitting results induced by Killing vector fields.

Inspired by Petersen and Wylie's work [82], in Chapter 2, we will study the isometry group $\mathrm{Iso}(M)$ and its Lie algebra of an irreducible non-trivial gradient Ricci soliton (M, g, f). Recall that a Riemannian manifold is said to be irreducible if no finite cover of it can be expressed (in the isometric sense) as a direct product of manifolds of smaller dimensions.

Problem 1.1. *Find an upper bound on the dimension of the Lie algebra of Killing vector fields on an irreducible non-trivial gradient Ricci soliton, and classify the spaces where this maximal dimension is attained.*

1.2 Nonlinear parabolic equations and super geometric flows

Turning the framework of geometric flow theory, we now present the concept of super Ricci flow, which was originally introduced by McCann and Topping [70] from the perspective of optimal transport theory. A smooth manifold $(M, g(x,t))_{t\in I}$ is called a super Ricci flow if

$$\frac{\partial g}{\partial t} \geq -2\mathrm{Ric}. \tag{1.4}$$

In [70], the authors discovered that the monotonicity property of the Wasserstein distance along the heat flow characterizes supersolutions to the Ricci flow equation. Later, several other characterizations have been investigated via important geometric inequalities of manifolds, Bakry-Émery gradient estimates, and also convexity of Entropies (see [45, 64, 66, 96]). Drawing upon these characterizations, Sturm [96] expanded the concept of super Ricci flow to time-dependent (non-smooth) metric measure spaces. His work marks the beginning of a new exploration into super Ricci flow from the intriguing standpoint of metric measure geometry. Recently, Bamler [6] demonstrated that the space of super Ricci flows, when pointed in a suitable manner, is compact in a specific topology.

For each $k \in \mathbb{R}$, a time-dependent Riemannian manifold $(M, g(x,t))_{t\in I}$ is termed a k-super Ricci flow if it satisfies the following condition:

$$\frac{\partial g}{\partial t} + 2\,\mathrm{Ric} \geq 2kg, \tag{1.5}$$

which is a natural extension of the concept of super Ricci flow. Moreover, the k-super Ricci flow can be seen as a time-dependent version of the Riemannian manifold whose Ricci curvature is bounded from below by k. It is evident that the (0)-super Ricci flow is exactly the super Ricci flow. Besides, when the equality in (1.5) holds, $(M, g(x,t))_{t\in I}$ is called k-Ricci flow. A k-supper Ricci flow $(M, g(x,t))_{t\in I}$ is said to be ancient when $I = (-\infty, 0]$. This concept is an extension of the ancient Ricci flow, which is well-known for its significant impact on the study of singularities in Ricci flow analysis.

The reduced distance and reduced volume were first introduced by Perelman in his groundbreaking paper [78] as two key tools for analyzing the Ricci flow. Later, Ye proved several properties of Perelman's reduced distance and obtained some estimates for the reduced volume [112]. Besides, the applications of these properties in the analysis of the asymptotic limits of κ-solutions of the Ricci flow have been presented by Ye in the follow-up paper [113]. Recently, in noteworthy paper [57], Kunikawa and Sakurai obtained Liouville type theorems for harmonic maps under ancient super Ricci flow with controlled growth, approaching the topic from Perelman's reduced geometric perspective. This paper is the continuation of a work with the same scope for functions in [56].

The next chapter of this thesis is also motivated from a work due to Ma [67]. In [67], for some constants a, b, the author considered the following nonlinear elliptic equation

$$\Delta u + au\ln u + bu = 0 \tag{1.6}$$

in a complete noncompact Riemannian manifold. From Ma's observation in [67], we know that the above equation is closely related to the equation (1.3) of the gradient Ricci soliton (M, g, f). Indeed, taking the trace of the equality (1.3), we deduce that

$$S + \Delta f = n\lambda.$$

Here S is the scalar curvature of M and n is the dimension of M. According to Proposition 2.3 in Chapter 2, we get

$$|\nabla f|^2 + S - 2\lambda f = A_0,$$

where A_0 is a constant. Combining the two above equations, we have

$$|\nabla f|^2 - \Delta f - 2\lambda f + n\lambda - A_0 = 0.$$

If we set $u = e^{-f}$, then by a simple computation, it follows that u solves

$$\Delta u + 2\lambda u\ln u + (n\lambda - A_0)\, u = 0. \tag{1.7}$$

Clearly, the above equation is a special case of the equation (1.6). Moreover, the equation (1.6) is naturally linked to geometric and functional inequalities on manifolds, particularly the logarithmic Sobolev inequality [105] and Perelman's $\mathcal{W}$-entropy [78]. Replacing u by $e^{\frac{b}{a}}u$, we see that the equation (1.6) is equivalent to the following equation

$$\Delta u + au \ln u = 0. \tag{1.8}$$

Inspired by the works of Kunikawa, Sakurai, and Ma, in Chapter 3, we will study gradient estimates for positive bounded solutions to the parabolic counterpart of equation (1.8) along ancient k-super Ricci flow and explore some of its applications. Specifically, we are interested in the following problem.

Problem 1.2. *Establish gradient estimates and Liouville type results for positive bounded solutions of the nonlinear parabolic equation related to Perelman's reduced distance*

$$\frac{\partial}{\partial t}u(x,t) = \Delta u(x,t) + au(x,t)\ln u(x,t) \tag{1.9}$$

along ancient k-super Ricci flow, where $a \in \mathbb{R}$.

A smooth metric measure space, also known as a weighted manifold or a manifold with density, can be viewed as a natural generalization of gradient Ricci solitons. Since Perelman's works [78, 79], this space has been the subject of extensive study by many mathematicians worldwide. Recall that a smooth metric measure space is a triple $(M, g, e^{-f}d\mu)$, where (M, g) is a complete Riemannian manifold of dimension $n \geq 3$ endowed with a weighted measure $e^{-f}d\mu$ for some $f \in \mathcal{C}^\infty(M)$ and $d\mu$ is the standard Riemannian volume measure of metric g. On $(M, g, e^{-f}d\mu)$, the weighted Laplacian Δ_f is defined by

$$\Delta_f \cdot := \Delta \cdot - \langle \nabla f, \nabla \cdot \rangle,$$

which is a natural generalization of the Laplace-Beltrami operator Δ to the smooth metric measure space context, and it coincides with the latter precisely when the potential f is a constant function. For any real number $m \geq 0$, the m-Bakry-Émery curvature is defined by

$$\mathrm{Ric}_f^m := \mathrm{Ric} + \mathrm{Hess} f - \frac{1}{m} df \otimes df.$$

When $m = 0$, it means that f is constant and Ric_f^m becomes the usual Ricci curvature Ric. When $m \to \infty$, we have the (∞-)Bakry-Émery Ricci curvature

$$\mathrm{Ric}_f := \mathrm{Ric}_f^\infty = \mathrm{Ric} + \mathrm{Hess} f.$$

It is not difficult to see that $\mathrm{Ric}_f^m \geq c$ infers $\mathrm{Ric}_f \geq c$, but the contrary may not be accurateaccurate. When Ric_f is bounded from below, many geometric properties of manifolds with the Ricci tensor bounded from below were also possibly extended to smooth metric measure spaces, but some extra assumptions on f are required; see [63, 103] for detailed discussion.

Motivated by the above works of Hamilton, McCann-Topping, and Perelman's work for the modified Ricci flow (see [66, 78], this flow is often referred to as the Perelman-Ricci flow), X.-D. Li et al. [64, 65] introduced the concept (k, m)-super Perelman-Ricci flow on manifolds equipped with time-dependent metrics and potentials. For $k, m \in \mathbb{R}$ and $m \geq 0$, a time-dependent smooth metric measure space $\left(M, g(x,t), e^{-f(x,t)} d\mu\right)_{t \in I}$ is called (k, m)-super Perelman-Ricci flow if

$$\frac{\partial g}{\partial t} + 2\,\mathrm{Ric}_f^m \geq -2kg. \tag{1.10}$$

It is worth noting that this flow is the weighted version of the k-super Ricci flow (1.5). Moreover, the (k, m)-super Perelman-Ricci flow is equivalent to the so-called curvature-dimension condition $\mathrm{CD}(k, m)$ in the sense of Sturm [95] and Lott-Villani [59]. When $m \to \infty$, i.e., if the metric $g(x,t)$ and the potential function $f(x,t)$ satisfy the following inequality

$$\frac{\partial g}{\partial t} + 2\mathrm{Ric}_f \geq -2kg, \tag{1.11}$$

we call $\left(M, g(x,t), e^{-f(x,t)} d\mu\right)_{t \in I}$ a (k, ∞)-super Perelman-Ricci flow, which can be viewed as a natural extended of the modified Ricci flow [78].

One of the most studied topics in geometric analysis during the 20th century is the Yamabe problem, introduced by Yamabe in his notable posthumous publication [108]. Let (M, g) be an n-dimensional smooth, compact Riemannian manifold with $n \geq 3$. The Yamabe problem can be viewed as a generalization of the Poincaré-Köbe uniformization theorem, which is a state that determines a constant scalar curvature metric $\widetilde{g}$ that is pointwise conformally related to g. Recall that the conformal class of g is defined to be $[g] = \left\{\widetilde{g} = u^{\frac{4}{n-2}} g : u \in C^\infty(M), u > 0\right\}$.

Then the scalar curvature $S_{\widetilde{g}}$ of the conformal metric $\widetilde{g}$ can be written as

$$S_{\widetilde{g}} = -\frac{4(n-1)}{n-2} u^{-\frac{n+2}{n-2}} \left(\Delta u - \frac{n-2}{4(n-1)} S_g u \right).$$

Here S_g is the scalar curvatures of g and Δ is the Laplace-Beltrami operator associated with g. From this observation, we see that the Yamabe problem amounts to find a positive solution u of the Yamabe equation

$$\Delta u - \frac{n-2}{4(n-1)} S_g u + \frac{n-2}{4(n-1)} S_{\widetilde{g}} u^{\frac{n+2}{n-2}} = 0 \tag{1.12}$$

where $S_{\widetilde{g}}$ is constant. This was resolved through the contributions of N. Trudinger [100], T. Aubin [4], and R. Schoen [85]. Their proofs utilize results from the calculus of variations and elliptic theory; for further details, refer to the survey article by Lee and Parker [60].

The Yamabe flow was initially explored by Hamilton in the unpublished work [42] as a tool for addressing the Yamabe problem. An n-dimensional manifold $(M, g(x,t))_{t \in I}$ equipped with a time-dependent metric is referred to as a Yamabe flow when it satisfies the following equation

$$\frac{\partial g}{\partial t} = -Sg, \tag{1.13}$$

where S is the scalar curvatures of the metric g. In [25], Chow studied the normalized Yamabe flow and demonstrated that this flow converges to a metric with constant scalar curvature. By assuming only that the initial metric is locally conformally flat, Ye established the convergence of the Yamabe flow [111], thereby improving upon Chow's result [25]. The scenario of metrics that are not conformally flat has been studied in a series of papers by Schwetlick and Struwe [87] and subsequently by Brendle [13, 14].

Inspired by the work presented in Chapter 3 and the advancements made in the smooth metric spaces discussed earlier, Chapter 4 will investigate the following problem.

Problem 1.3. *Study some analytical aspects of a general type of nonlinear parabolic equation concerning the weighted Laplacian*

$$\left(\frac{\partial}{\partial t} - a(x,t) - \Delta_f \right) u(x,t) = F(u(x,t)) \tag{1.14}$$

on a smooth metric measure space with the metric evolving under the (k,∞)*-super Perelman-Ricci flow* (1.11) *and the Yamabe flow* (1.13)*, where* $a(x,t)$ *is a function which is* $\mathcal{C}^2$ *in the* x*-variable and* $\mathcal{C}^1$ *in the* t*-variable, and* $F(u)$ *is a* $\mathcal{C}^2$ *function of* u*.*

The nonlinear heat equation (1.14) has garnered significant attention from mathematicians due to its applications in mathematics, physics, and various other fields. In the case $F(u) = 0$ and f is constant, the equation (1.14) reduces to the Schrödinger equation, which is one of the fundamental equations in quantum mechanics. When $F(u) = bu\ln u$ for $b \in \mathbb{R}$ and $a(x,t) \equiv 0$, the equation (1.14) is precisely the equation closely related to the Ricci gradient soliton (1.8). When u is a stationay solution (namely $u_t \equiv 0$), f is constant and $F(u) = bu^\alpha$, the equation (1.14) becomes the Yamabe type equation (1.12). Furthermore, when $F(u) = bu^\alpha + cu^\beta$, the equation (1.14) is closely associated with the Lichnerowicz-type equations for Einstein-scalar fields, which are a key area of research in Einstein scalar field theory within general relativity [11, 77]. In general, the nonlinear parabolic equation (1.14) is referred to as a weighted reaction-diffusion equation, which appears in various mathematical models across physics, chemistry, and biology (see [92]), where $au + F(u)$ and $\Delta_f u$ are the reaction term and the diffusion term, respectively.

1.3 Translating solitons of the mean curvature flow

The last part of this thesis shifts the focus to issues related to mean curvature flows. One of the primary motivations for mean curvature flow comes from geometric applications, akin to the Ricci flow of metrics on Riemannian manifolds. This flow is a powerful tool for obtaining classification results for hypersurfaces that meet specific curvature conditions, deriving isoperimetric inequalities, and producing minimal surfaces. Besides, mean curvature flow is pivotal in describing the evolution of interfaces in various multiphase physical models (see, e.g., [75, 93]), and its origin can be traced back to Mullins' influential paper [75]. This relevance stems from its characteristic as a gradient-like flow of the area functional, making it inherently applicable to problems involving surface energy (see [69]).

We now recall the definition of mean curvature flow. Let $X : M^n \to \mathbb{R}^{n+m}$ be a smooth immersion of an n-dimensional smooth manifold in Euclidean space $\mathbb{R}^{n+m}$. A smooth one-parameter family $X_t = X(\cdot, t)$ of immersions $X_t : M \times [0,T) \to \mathbb{R}^{m+n}$ with corresponding images $M_t = X_t(M)$ is called the mean curvature flow for a submanifold M in $\mathbb{R}^{m+n}$ if it satisfies the following condition

$$\begin{cases} \frac{d}{dt}X(x,t) = H(x,t), \\ X(x,0) = X(x), \end{cases} \tag{1.15}$$

for any $(x,t) \in M \times [0,T)$, where $H(x,t)$ is the mean curvature vector of M_t at $X_t(x)$ in $\mathbb{R}^{m+n}$.

One of the key aspects of studying mean curvature flow is the analysis of singularities. In various scenarios, the second fundamental form with respect to the family M_t may experience singularities. For instance, if M is compact, the second fundamental form will blow up in a finite time. Based on the blow-up rate of the second fundamental form, we categorize the singularities of mean curvature flow into two types: Type-I singularities and Type-II singularities. The geometry of the solution near Type-II singularities is more challenging to control, making the study of Type-II singularities significantly more complex than that of Type-I singularities.

A solution to (1.15) is said to be a translating soliton (or simply a translator) if there exists a constant vector V with unit length in $\mathbb{R}^{n+m}$ such that

$$H = V^{\perp}, \tag{1.16}$$

where $V^{\perp}$ denotes the normal component of V in $\mathbb{R}^{n+m}$. Translating solitons are significant in the theory of mean curvature flow because they arise as blow-up solutions at type II singularities. On the other hand, every translating soliton is a special solution that moves only in a constant direction V without deforming its shape under the mean curvature flow, specifically, the solution is given by $M_t = M + tV$. There are few examples of translating solitons even in the hypersurface case. The primary examples are those translating solitons that are also minimal hypersurfaces. Indeed, by (1.16) we know that V must be tangential to the translator. Consequently, these solitons could have the form of $\widetilde{M} \times L$, where L is a line parallel to V and $\widetilde{M}$ is a minimal hypersurface in $L^{\perp}$. We can find more translating solitons, for examples in [48, 51] and the references therein.

Through examples of translating solitons, we can derive interesting results and establish a framework for their classification. Recently, in [106], Xin studied various geometric aspects of translating solitons, including volume growth, the generalized maximum principle, Gauss maps, and certain functions related to the Gauss map. In addition, he provided integral estimates for the squared norm of the second fundamental form. Using these results, Xin demonstrated a rigidity theorem for translators in the Euclidean space in higher codimensions. Some of Xin's results were subsequently extended by Wang, Xu, and Zhao by using integral curvature pinching conditions of the trace-free second fundamental form (see [101]).

Utilizing the approach direction from the theory of weighted minimal hypersurfaces, in the papers [48, 49], Impera and Rimoldi studied the topological structure at infinity of translating solitons of the mean curvature flow. In particular, they established weighted Sobolev inequalities and utilized these results to demonstrate that an f-stable translator can have at most one end. Additionally, they explored the relationship between the space of L^2-weighted harmonic 1-forms, cohomology with compact support, and the index of the translator in terms of the generalized Morse index of a stable operator. Building on the Sobolev inequalities established by Impera and Rimoldi, Kunikawa and Sato [55] noted that any complete f-stable translating soliton does not allow for any codimension one cycle. Consequently, any two-dimensional complete f-stable translator must have genus zero.

Inspired by the research results on translating solitons mentioned above, in Chapter 5 of this thesis, we are interested in the following problem.

Problem 1.4. *Study of the rigidity properties and connectedness at infinity of complete translating solitons in the Euclidean space via the second fundamental form.*

1.4 Structure of the present work

As mentioned earlier, the dissertation is divided into five chapters. In addition to Chapter 1, the remaining four chapters will be described below. It also includes a section listing the author's related papers, a Conclusions section, and a list of references. Below is a brief overview of the contents of each chapter, from Chapter 2 to Chapter 5.

In Chapter 2 of this dissertation, we investigate the isometry group $\mathrm{Iso}(M)$ and its Lie algebra of an irreducible non-trivial gradient Ricci soliton (M, g, f). This chapter aims to study Problem 1.1, which is based on the paper to appear in *Forum Mathematicum*, `https://doi.org/10.1515/forum-2024-0325`.

Chapter 3 of this dissertation is devoted to studying the nonlinear parabolic equation (1.9) related to Perelman's reduced distance, along ancient k-super Ricci flow. This chapter aims to study Problem 1.2, which is based on the paper [32] published in the *Journal of Mathematical Analysis and Applications*.

In Chapter 4 of this dissertation, we focus instead on studying the general type of nonlinear parabolic equation (1.14) on a smooth metric measure space with the metric evolving under the (k, ∞)-super Perelman-Ricci flow (1.11) and the Yamabe flow (1.13). Chapter 4 aims to study Problem 1.3, based on the paper [34] published in *Nonlinear Analysis*.

Chapter 5 of this dissertation focuses on studying some aspects of complete translating solitons in the Euclidean space. Chapter 5 aims to study Problem 1.4, which is based on the paper [35] published in *Manuscripta Mathematica*.

The results of this dissertation were presented at

- The weekly seminar of Geometric Analysis group (June 28, 2023, Vietnam Institute for Advanced Studies in Mathematics, Hanoi);

- The monthly seminar of the Department of Geometry, (December 12, 2023, Hanoi National University of Education, Hanoi);

- The 10th Vietnam Mathematical Congress, Committee on Partial Differential Equations (August 11, 2023, the University of Da Nang-University of Science and Education, Da Nang);

- The Workshop "Some selected topics in Geometric Analysis and applications" (February 1, 2024, Hanoi University of Civil Engineering, Hanoi).

Chapter 2

On isometry groups of gradient Ricci solitons

This chapter is written based on the paper "Ha Tuan Dung, Hung Tran (2025), On isometry groups of gradient Ricci solitons, to appear in *Forum Mathematicum*, https://doi.org/10.1515/forum-2024-0325" [35] and focuses on examining Problem 1.1 discussed in Chapter 1. We specifically investigate the isometry group and its Lie algebra of an irreducible, non-trivial gradient Ricci soliton (M, g, f). Our goal is to determine the maximum dimension of the isometry group and study the structure of this manifold when the maximal dimension is attained. Towards that end, we recall the Lie algebra of the isometry group of (M, g, f):

$$\mathfrak{iso}(M, g) := \{X \text{ is a smooth tangent vector field on } M, \mathcal{L}_X g = 0\}.$$

Closely related to the Lie algebra $\mathfrak{iso}(M, g)$ is the Lie algebra of Killing vector fields preserving f:

$$\mathfrak{iso}_f(M, g, f) := \{X \text{ is a smooth tangent vector field on } M, \mathcal{L}_X g = 0 = \mathcal{L}_X f\}.$$

Throughout this chapter, for convenience in presentation, we will abbreviate the term gradient Ricci soliton as GRS.

In order to achieve the main goal, we first give a result estimating the dimension of $\mathfrak{iso}_f(M, g, f)$ and classify the spaces where this maximal dimension is achieved.

Theorem 2.1. *Let (M^n, g, f), with $n \geq 3$, be a GRS. If f is non-constant then $\mathfrak{iso}_f(M, g, f)$ is of dimension at most $\frac{1}{2}(n-1)n$ and equality happens iff each connected component of a regular level set of f is a space of constant curvature.*

Let $(\mathbb{N}^{n-1}, g_{\mathbb{N}})$ denote the space form model. If $g_{\mathbb{N}}$ is non-flat, the equality happens iff the metric is locally a warped product. That is, there is an open dense *subset such that around each of its points, there is a neighborhood diffeomorphic to a product $I \times \mathbb{N}$ and the metric g is given by $g = dt^2 + F^2(t)g_{\mathbb{N}}$. Here, I is an open interval, and $F : I \mapsto \mathbb{R}^+$ is a smooth function.*

Furthermore, it is possible to relax the assumption on preserving f. A Riemannian manifold is locally irreducible if it is not a local Riemannian product metric around each point.

Theorem 2.2. *Let (M^n, g, f), with $n \geq 3$, be a locally irreducible non-trivial GRS. Then $\mathfrak{iso}(M, g)$ is of dimension at most $\frac{1}{2}(n-1)n$. In addition, equality happens iff it is smoothly constructed, as in the case of equality of Theorem 2.1.*

The above theorems are essentially local. That is, there is no mention of the completeness of the metric. Indeed, the soliton structure is so rigid that it is difficult to complete the above metrics.

Theorem 2.3. *Let (M^n, g, f), with $n \geq 3$, be an irreducible non-trivial complete GRS. Then $\mathfrak{iso}(M, g)$ is of dimension at most $\frac{1}{2}(n-1)n$. For $\lambda \geq 0$, equality happens iff $\lambda = 0$ and it is isometric to a Bryant soliton.*

Chapter 2 is organized as follows. In Section 2.1, we recall basic notations and collect preliminary materials that we will use in the rest of this chapter. The main results will be proved in Section 2.2. Finally the Appendix considers the case that each level set of a GRS is Euclidean.

2.1 Preliminaries

This section is to recall auxiliary results on Killing vector fields, group actions on manifolds, and gradient Ricci solitons. The main references are [2, 26, 52, 53, 82, 83].

2.1.1 Killing vector fields and group actions on manifolds

In this subsection, we briefly review basic properties of Killing vector fields and their relationship to the isometry group. Besides, we also recall some basic concepts related to group actions on manifolds. The standard texts are [2, 52, 83]. We begin by providing the definition of Riemannian isometries.

Definition 2.1. *Let (M, g_M) and (N, g_N) be Riemannian manifolds. An isometry* *from M to N is a diffeomorphism $\phi : M \to N$ such that*

$$\phi^* (g_N) = g_M.$$

In other words, ϕ is an isometry if for all $p \in M$ and tangent vectors $X_p, Y_p \in T_pM$,

$$g_M|_p (X_p, Y_p) = g_N|_{\phi(p)} ((d\phi)_p (X_p), (d\phi)_p (Y_p)).$$

In this sense, we say that ϕ preserves the metric structure. In addition, M and N are called isometric.

The set of all isometries of a Riemannian manifold (M, g) onto itself forms a group (indeed a Lie group), which is denoted by $\mathrm{Iso}(M)$ and called the isometry group of M.

Definition 2.2. *A vector field X on a Riemannian manifold (M, g) is called a Killing vector field if the Lie derivative with respect to X of the metric g vanishes, i.e., $\mathcal{L}_X g = 0$.*

The following proposition shows the relationship between Killing vector fields and isometries. For a proof, we refer the reader to [83, Proposition 8.1.1].

Proposition 2.1. *A vector field X on a Riemannian manifold (M, g) is a Killing vector if and only if the local flows generated by X act by isometries.*

Because of Proposition 2.1, Killing vector fields are also commonly known as *infinitesimal isometries*, a terminology that arises from the idea of integrating vector fields to obtain isometries. Furthermore, they enjoy strong analytic properties.

Proposition 2.2. [83, Proposition 8.1.4] *Let X be a Killing vector field on a Riemannian manifold (M, g). If there exists a point $p \in M$ such that $X_p = 0$ and $(\nabla X)_p = 0$, then X is identical* 0.

Remark 2.1. The set of all Killing vector fields on a Riemannian manifold (M, g) is a Lie algebra, and denoted by $\mathfrak{iso}(M, g)$. Furthermore, by Theorem 8.1.6 in [83], if the Levi-Civita connection induced by the Riemannian metric g on M is complete, then so is each Killing vector field. In that case, $\mathfrak{iso}(M, g)$ is the Lie algebra of $\mathrm{Iso}(M)$.

Next, we recall a result estimating the dimension of the Lie algebra $\mathfrak{iso}(M, g)$ and $\mathrm{Iso}(M, g)$, which will play an important role in our proof of Theorem 2.1.

Lemma 2.1. [83, Theorem 8.1.6] [52, Theorem 1, Note 10] *Let (M, g) be a connected Riemannian manifold of dimension n. Then the Lie algebra $\mathfrak{iso}(M, g)$ is* *of dimension at most* $\frac{1}{2}n(n+1)$. *If* $\dim \mathfrak{iso}(M,g) = \frac{1}{2}n(n+1)$, *then* M *is a space of constant curvature. Furthermore, if* $\dim \mathrm{Iso}(M) = \frac{1}{2}n(n+1)$, *then* M *is isometric to one of the following:*

(i) *an* n*-dimensional Euclidean space* $\mathbb{R}^n$,

(ii) *an* n*-dimensional sphere* $\mathbb{S}^n$,

(iii) *an* n*-dimensional real projective space,*

(iv) *an* n*-dimensional, simply connected hyperbolic space.*

In the rest of this subsection, we recall some basic notions about group actions on manifolds following the book by Alexandrino and Bettiol [2].

Definition 2.3. *Let* G *be a Lie group and* M *a smooth manifold. A smooth map* $l : G \times M \to M$ *is called a (left) action of* G *on* M, *or a (left)* G*-action on* M, *if*

(i) $l(e,x) = x$, *for all* $x \in M$, *where* e *is the identity element of* G;

(ii) $l(\mathsf{g}_1, l(\mathsf{g}_2, x)) = l(\mathsf{g}_1\mathsf{g}_2, x)$, *for all* $\mathsf{g}_1, \mathsf{g}_2 \in G$ *and* $x \in M$.

We often write $\mathsf{g} \cdot x$ or just $\mathsf{g}x$ in place of the more pedantic notation $l(\mathsf{g}, x)$. A right action $r : M \times G \to M$ can be defined analogously and we write $x \cdot \mathsf{g}$ or $x\mathsf{g}$.

Definition 2.4. *An action is said to be proper if the associated map* $G \times M \mapsto M \times M$, *given by*

$$G \times M \ni (\mathsf{g}, x) \longmapsto (\mathsf{g} \cdot x, x) \in M \times M \tag{2.1}$$

is proper, i.e., if the preimage of any compact subset of $M \times M$ *under* (2.1) *is a compact subset of* $G \times M$.

From Proposition 3.62 and Theorem 3.65 in [2], we see that actions by closed subgroups of isometries are proper, and conversely every proper action can be made isometric with respect to a certain Riemannian metric.

Definition 2.5. *A Riemannian manifold* (M,g) *is said to be homogeneous if its isometry group acts transitively, i.e., for each pair of points* $x, y \in M$ *there is a* $\mathsf{g} \in \mathrm{Iso}(M)$ *such that* $\mathsf{g} \cdot x = y$.

2.1.2 Some basis results on gradient Ricci solitons

In this subsection, we shall recall some basic facts and collect preliminaries about GRS. Then, the Let (M, g, f) be a GRS of dimension $n \geq 3$. Then the smooth potential function $f : M \to \mathbb{R}$ satisfies the following equation

$$\operatorname{Ric} + \operatorname{Hess} f = \lambda g, \tag{2.2}$$

where $\lambda \in \mathbb{R}$, Ric is the Ricci curvature of M and Hess f denotes the Hessian of f. The quantities Ric, f, and the scalar curvature S of M are related by the following equations [39, Proposition 2.1].

Proposition 2.3. *For any gradient Ricci soliton (M, g, f), we have*

$$\Delta_f S + 2|\operatorname{Ric}|^2 = 2\lambda S, \tag{2.3}$$

$$S + |\nabla f|^2 - 2\lambda f = C \tag{2.4}$$

for some constant C. Here Δ_f denotes the f-Laplacian, $\Delta_f \cdot := \Delta \cdot - \langle \nabla f, \nabla \cdot \rangle$.

In [82, Proposition 2.1], Petersen and Wylie proved the following result about a Killing field on a GRS.

Proposition 2.4. *If X is a Killing field on a gradient Ricci soliton (M, g, f), then $\nabla(Xf)$ is parallel. Moreover, if $\lambda \neq 0$ and $\nabla(Xf) = 0$, then also $Xf = 0$.*

Remark 2.2. We emphasize that to prove the above result, Petersen and Wylie used the condition that scalar curvature is bounded. However, such an assumption can be omitted since, for $\lambda \neq 0$, the scalar curvature of a GRS is always bounded from either below or above [3, Theorem 8.6]). This is enough for the argument to go through.

Consequently, Petersen and Wylie [82] gave the following splitting result.

Lemma 2.2. [82, Corollary 2.2] *If X is a Killing field on a GRS $(M, g, f,)$ then either $\nabla(Xf) = 0$ or M locally splits a line isometrically. The latter means that around each point p, there is a neighborhood $U = V \times I$, where V is an open neighborhood of a submanifold and $T_pV \perp (\nabla(Xf))_p$ and I is an open interval. The Riemannian metric in U is the direct product of the induced metrics on each factor, and $(V, g_{|V}, f)$ is a GRS.*

For (M, g, f) a shrinking gradient Ricci soliton, upon scaling the metric g by a constant, we can assume that $\lambda = \frac{1}{2}$. Then the equation (2.2) takes the form

$$\mathrm{Ric} + \nabla^2 f = \frac{1}{2} g. \tag{2.5}$$

By adding a constant to f if necessary and the equation (2.4), we may normalize the soliton such that

$$S + |\nabla f|^2 = f. \tag{2.6}$$

Moreover, according to a result by Chen [22, Corollary 2.5] (see also [3, Theorem 8.6]), we have $S \geq 0$ for any shrinking gradient Ricci soliton. This and (2.6) entail that $f \geq 0$. On the other hand, from Haslhofer-Müller's works [44, Lemma 2.1] (see also [18, Theorem 1.1]), we know that the potential function f has quadratic growth at infinity. Using these results, we obtain the following proposition.

Proposition 2.5. *Let (M, g, f) be an n-dimensional complete noncompact shrinking gradient Ricci soliton with (2.5) and (2.6). Then, each regular level set of f is a compact set.*

Proof. For each regular value $c \in f(M)$, we consider the level set M_c of f. Since f is a smooth function and $\{c\}$ is a closed set, $M_c = f^{-1}(c)$ is also a closed set. By Lemma 2.1 in [44], there exists a point $p \in M$ where f attains its infimum and f satisfies the following quadratic growth estimate

$$\frac{1}{4}\left[(r(x) - 5n)_+\right]^2 \leq f(x) \leq \frac{1}{4}\left(r(x) + \sqrt{2n}\right)^2,$$

where $r(x)$ is a distance function from p to x, and $a_+ = \max\{a, 0\}$ for $a \in \mathbb{R}$. This and the fact that $f \geq 0$ imply that M_c is a bounded set and, therefore, M_c is a compact set. □

2.2 Dimension bound and Rigidity

This section is devoted to the proof of our main results. Let (M, g, f) be a GRS of dimension $n \geq 3$. Recall

$$\mathfrak{iso}(M, g) := \{X \text{ is a smooth tangent vector field on } M, \mathcal{L}_X g = 0\}.$$

We also define

$$\mathfrak{iso}_f(M, g, f) := \{X \text{ is a smooth tangent vector field on } M, \mathcal{L}_X g = 0 = \mathcal{L}_X f\}. \tag{2.7}$$

Then, we see that $\mathfrak{iso}_f(M,g,f) \subset \mathfrak{iso}\,(M,g)$ is a vector subspace. Towards our goal, we will establish the following lemma concerning $\mathfrak{iso}\,(M,g)$.

Lemma 2.3. *If $X \in \mathfrak{iso}(M,g)$ and $g(X,\nabla f) = Xf$ is constant then*

$$[X, \nabla f] = 0.$$

Proof. We observe that

$$\begin{aligned} g\left(\mathcal{L}_X \nabla f, Y\right) &= g\left(\nabla_X \nabla f - \nabla_{\nabla f} X, Y\right) \\ &= (\text{Hess}\, f)(X,Y) + g\left(\nabla_Y X, \nabla f\right) - \left(\mathcal{L}_X g\right)(Y, \nabla f) \\ &= (\text{Hess}\, f)(X,Y) - g\left(X, \nabla_Y \nabla f\right) + Y\left(\mathcal{L}_X f\right) - \left(\mathcal{L}_X g\right)(Y, \nabla f) \\ &= Y\left(\mathcal{L}_X f\right) - \left(\mathcal{L}_X g\right)(Y, \nabla f) \end{aligned} \tag{2.8}$$

for any $Y \in TM$. Since X is a Killing vector field, $\left(\mathcal{L}_X g\right)(Y,\nabla f) = 0$. Since $\mathcal{L}_X f = Xf$ is a constant, $Y\left(\mathcal{L}_X f\right) = 0$. Combining these results yields

$$[X, \nabla f] = \mathcal{L}_X \nabla f = 0.$$

The proof is complete. □

We now give the proof of Theorem 2.1.

Proof of Theorem 2.1. Let M_c be a level set of f with the induced metric $g_c := g_{|TM_c}$, where $c \in f(M)$ is a regular value. By the level set theorem [99], (M_c, g_c) is a smooth submanifold of co-dimension one. Consider $X \in \mathfrak{iso}_f(M,g,f)$ and let φ_t^X denote the local flow generated by the vector field X. Then, we have

$$X = \left.\frac{d}{dt}\right|_{t=0} \varphi_t^X.$$

Since $\mathcal{L}_X g = 0$ and $\mathcal{L}_X f = 0$, we deduce that $\left(\varphi_t^X\right)^* g = g$ and

$$\left(\varphi_t^X\right)^* f = f \Leftrightarrow f \circ \varphi_t^X = f, \tag{2.9}$$

where $\left(\varphi_t^X\right)^*$ is the pull-back of φ_t^X. By Proposition 2.1, we see that $\varphi_t^X : M \to M$ generates local isometries and $\varphi_t^X\left(M_c\right) \subseteq M_c$. From this, we notice that φ_t^X induces a map $\widetilde{\varphi}_t^X \equiv \varphi_t^X|_{M_c} : M_c \to M_c$. We consider the vector field

$$\widetilde{X} = X|_{M_c} = \left.\frac{d}{dt}\right|_{t=0} \left(\varphi_t^X|_{M_c}\right) = \left.\frac{d}{dt}\right|_{t=0} \widetilde{\varphi}_t^X.$$

Since $\widetilde{\varphi}_t^X$ is an isometry on M_c, we conclude that

$$\widetilde{X} \in \mathfrak{iso}\,(M_c, g_c) = \{X \in TM_c \mid \mathcal{L}_X g_c = 0\}\,.$$

Thus, the map

$$\pi : \mathfrak{iso}_f(M, g, f) \to \mathfrak{iso}\,(M_c, g_c)$$
$$X \mapsto \pi(X) := \widetilde{X} = X\Big|_{M_c}$$

is well-defined. Moreover, π is a linear map. Next, we will prove that π is injective. Suppose that $X|_{M_c} \equiv 0$, where $X \in \mathfrak{iso}_f(M, g, f)$. Since $\mathcal{L}_X g = 0$ and $\mathcal{L}_X f = 0$, Lemma 2.3 yields

$$[X, \nabla f] = \mathcal{L}_X \nabla f = 0. \tag{2.10}$$

Let $p \in M_c$ and $Y \in TM$ be an arbitrary vector field. Then, we have $Y = Z + W$, where $Z \in TM_c$ and $W \in T^{\perp}M_c$. Since ∇f is a normal vector field of TM_c, $W = \eta \nabla f$, where η is a smooth function. Therefore, we get

$$(\nabla_Y X)|_p = (\nabla_{Z+W} X)|_p = (\nabla_Z X)|_p + \eta\,(\nabla_{\nabla f} X)|_p = \eta\,(\nabla_{\nabla f} X)|_p\,. \tag{2.11}$$

The last equality follows from $X|_{M_c} = 0$. Furthermore, using (2.10), we compute

$$(\nabla_{\nabla f} X)|_p = (-[X, \nabla f] + \nabla_X \nabla f)|_p = (\nabla_X \nabla f)|_p = 0. \tag{2.12}$$

Since $Y \in TM$ is an arbitrary vector field, we conclude that $(\nabla X)|_p = 0$. Since $X_p = 0$, by Proposition 2.1, we deduce that $X \equiv 0$. This shows that the map π is injective. From Lemma 2.1 and note that $\dim M_c = n - 1$, we obtain

$$\dim \mathfrak{iso}_f(M, g, f) \le \dim \mathfrak{iso}\,(M_c, g_c) \le \frac{1}{2}(n-1)n. \tag{2.13}$$

Next, we will consider the case $\dim \mathfrak{iso}_f(M, g, f) = \frac{1}{2}(n-1)n$. By Lemma 2.1, each regular connected component of f with the induced metric must be of constant curvature. Consequently, each is homogeneous and complete [52, Theorem IV.4.5]. Thus, the Lie algebra $\mathfrak{iso}_f(M, g, f)$ indeed generates a global group of isometries on (M, g), and the action is transitive on each regular level set. Therefore, S is constant on each regular level set and, by Proposition 2.3, so is $|\nabla f|$ and $\frac{df}{|\nabla f|}$ is closed and locally exact. Define t by $dt = \frac{df}{|\nabla f|}$ then the metric can be written locally as

$$g = dt^2 + g_t,$$

where g_t is a family of metrics on the differentiable manifold corresponding to a regular connected component. Let L denote the shape operator and

$$\nu := \frac{\partial g_t}{\partial t} = 2g_t \circ L.$$

Furthermore, by the constancy of $|\nabla f|$ on each regular connected component, singular values for $f : M \mapsto \mathbb{R}$ are isolated. By continuity, nearby connected components must be obtained from the same model space $(\mathbb{N}^{n-1}, g_{\mathbb{N}})$.

Since g_t is homogeneous, so is ν, and it suffices to consider its value at a point. We recall the evolution of the Ricci tensor, $\mathrm{Ric}_t := \mathrm{Ric}(g_t)$, [26, page 109], for normal coordinates,

$$\begin{aligned}
\frac{\partial}{\partial t}\mathrm{Ric}_{ij} &= -\frac{1}{2}\Big(\Delta_L \nu_{ij} + \nabla_i\nabla_j \operatorname{trace}(\nu)) - \nabla_i(\delta\nu)_j - \nabla_j(\delta v)_i\Big),\\
\Delta_L \nu_{ij} &= \Delta \nu_{ij} + 2\,\mathrm{Rm}_{kijl}\,\nu_{kl} - \mathrm{Ric}_{ik}\,\nu_{jk} - \mathrm{Ric}_{jk}\,\nu_{ik},\\
\widehat{\mathrm{Rm}}\,(\nu)_{ij} &= 2\,\mathrm{Rm}_{kijl}\,\nu_{kl} - \mathrm{Ric}_{ik}\,\nu_{jk} - \mathrm{Ric}_{jk}\,\nu_{ik}.
\end{aligned}$$

As ν is homogeneous, all spacial derivatives vanish.

Claim. If $g_{\mathbb{N}}$ is non-flat then ν is a multiple of $g_{\mathbb{N}}$.

Proof of the claim. Since g_t is isomorphic to a space form, Ric is a multiple of the metric. Thus, Ric, when considered as a linear map on the tangent space, is a multiple of the identity for each t. Thus, so is its derivative. If $\mathrm{Rm}(g_{\mathbb{N}}) \neq 0$ then $\widehat{\mathrm{Rm}}(\nu)$ is a linear combination of a non-trivial multiple of ν and a multiple of the identity. The result then follows.

Thus, if $\mathrm{Rm}(g_{\mathbb{N}}) \neq 0$ there is a local diffeomorphism $\phi : \mathbb{N} \times I \mapsto U$, an open neighborhood in M, such that

$$\phi^*(g) = \phi^*(dt^2 + g_t) = dt^2 + F^2(t)\pi^* g_{\mathbb{N}}.$$

The result then follows. ☐

Remark 2.3. The case that $g_{\mathbb{N}}$ is flat means each level set is an Euclidean space. Their analysis will be carried out in the Appendix.

Next, we will apply Theorem 2.1 to prove Theorem 2.2.

Proof of Theorem 2.2. Since M is locally irreducible, by Lemma 2.2, $\nabla(Xf) \equiv 0$ for any Killing vector field $X \in \mathfrak{iso}(M, g)$. We then consider two possible cases.

Case 1: $\lambda \neq 0$. By Prop. 2.4, $Xf = 0$. That is, each Killing vector field automatically preserves f. Thus, $\mathfrak{iso}(M,g) \equiv \mathfrak{iso}_f(M,g,f)$ and the result then follows from Theorem 2.1.

Case 2: $\lambda = 0$. If the scalar curvature S of (M,g,f) is a constant, then from (2.3), we obtain $\text{Ric} \equiv 0$, and hence (M,g,f) is Ricci-flat, which is a contradiction to our non-triviality assumption. Thus, S is non-constant, and one observes that it is invariant under isometries. Hence

$$\begin{aligned}\mathfrak{iso}_S(M,g,f) :&= \{X \text{ is a smooth tangent vector field on } M, \mathcal{L}_X g = 0 = \mathcal{L}_X S\}\\ &= \mathfrak{iso}(M,g).\end{aligned}$$

Repeating the argument as in the proof of Theorem 2.1 we have, for M_c a regular level set of S,

$$\dim \mathfrak{iso}(M,g) = \dim \mathfrak{iso}_S(M,g,f) \leq \dim \mathfrak{iso}\,(M_c, g_c) \leq \frac{1}{2}(n-1)n.$$

If the equality happens then, by Lemma 2.1, each regular connected component of S with the induced metric must be of constant curvature. Furthermore, $|\nabla S|$ is also invariant by the isometric action, and the rest is verbatim as in the proof of Theorem 2.1. □

Proof of Theorem 2.3. First, by Theorem 2.2, $\dim \mathfrak{iso}(M,g) \leq \frac{1}{2}(n-1)n$ and equality happens only if each connected component of a regular level set of f is a space form $(\mathbb{N}^{n-1}, g_{\mathbb{N}})$. We now suppose that $\dim \mathfrak{iso}(M,g) = \frac{1}{2}(n-1)n$. Consequently, each regular level set is homogeneous and complete, and consequently, $\mathfrak{iso}(M,g)$ is the Lie algebra of the isometry group on each regular connected component. We will divide the rest of the proof into cases.

Case 1: $\lambda > 0$. By Proposition 2.5 each regular connected component is compact. Then, by Lemma 2.1, the model space $(\mathbb{N}, g_{\mathbb{N}})$ must be spherical (round sphere or the real projective space). Then, from Theorem 2.2, the Riemannian metric is a local warped product $g = dt^2 + F^2(t)g_{\mathbb{N}}$. By [12, Theorem 1], the Weyl tensor is vanishing, and (M,g) is locally conformally flat. The classification of a GRS with such property for $\lambda > 0$ is well-known. By [54, Theorem 1], (M,g,f) must be either the Gaussian shrinking gradient Ricci soliton on $\mathbb{R}^n$, the round cylinder shrinker on $\mathbb{S}^{n-1} \times \mathbb{R}$, or the round sphere shrinker on $\mathbb{S}^n$. They are all rigid.

Case 2: $\lambda = 0$. If the metric $g_{\mathbb{N}}$ is flat, that is (M^n, g, f) $(n \geq 3)$ is a steady gradient Ricci solition with local Euclidean level sets:

$$g := dt^2 + g_t = dt^2 + \sum_i h_i^2(t) dx_i^2,$$

where each function h_i is smooth, then from the system (2.19) in Appendix, we get

$$\begin{cases} u_j' = (u_0 - A)\, u_j \\ u_0' = B + (u_0 - A)\, A, \end{cases} \tag{2.14}$$

where $u_0 := f'$, $u_i := \frac{h_i'}{h_i}$ and

$$A := \sum_i \frac{h_i'}{h_i}, \quad B := \sum_i \left(\frac{h_i'}{h_i}\right)^2.$$

Note that $A' = (u_0 - A)\, A$. We rewrite the system (2.14) as by the first equation of the above system, we obtain

$$\frac{u_j'}{u_j} = \frac{u_i'}{u_i}$$

for all j, i. This implies that there is a smooth function h such that $\frac{u_j'}{u_j} = \frac{h'}{h}$ for all j. Then, we have

$$\left(\frac{u_j}{h}\right)' = \frac{u_j' h - u_j h'}{h^2} = 0.$$

Thus $u_j = a_j h$ for some constant a_j. From this and (2.14), one finds that

$$\begin{cases} h' = lh \\ l' = bh^2, \end{cases} \tag{2.15}$$

where

$$l = u_0 - ah, a = \sum_i a_i, b = \sum_i a_i^2.$$

One can notice that

$$\frac{\mathrm{d}l}{\mathrm{d}h} = \frac{\mathrm{d}l}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}h} = \frac{b}{l} h.$$

This implies that

$$\int l \mathrm{d}l = \int bh \mathrm{d}h.$$

Consequently,

$$bh^2 = l^2 + C$$

for some constant C. This and (2.15) lead to

$$l' = l^2 + C. \tag{2.16}$$

Now, we consider three possible cases.

Case 1: $C = 0$. Then the equation (2.16) becomes $l' = l^2$. Using this, we find that

$$l(t) = -\frac{1}{t + C_1}, \quad h(t) = \pm\frac{1}{\sqrt{b}\,(t + C_1)},$$

and

$$u_0(t) = l(t) + ah(t) = -\frac{1}{t + C_1} \pm \frac{a}{\sqrt{b}\,(t + C_1)},$$

for some constant C_1.

Case 2: $C > 0$. Then we set $C = D^2$ for some constant D and the equation (2.16) becomes $l' = l^2 + D^2$. Thus, we have

$$l(t) = D\tan\left(Dt + D_1\right), \quad h(t) = \pm\frac{D}{\sqrt{b}\cos\left(Dt + D_1\right)},$$

and

$$u_0(t) = l(t) + ah(t) = D\tan\left(Dt + D_1\right) \pm \frac{aD}{\sqrt{b}\cos\left(Dt + D_1\right)},$$

for some constant D_1.

Case 3: $C < 0$. Then we set $C = -D^2$ for some constant D and the equation (2.16) becomes $l' = l^2 - D^2$. From this, we get

$$l(t) = \frac{D\left(e^{2Dt} + D_1\right)}{e^{2Dt} - D_1}, \quad h(t) = \pm\frac{4D_1D^2e^{2Dt}}{\sqrt{b}\left(e^{2Dt} - D_1\right)},$$

and

$$u_0(t) = l(t) + ah(t) = \frac{D\left(e^{2Dt} + D_1\right)}{e^{2Dt} - D_1} \pm \frac{4aD_1D^2e^{2Dt}}{\sqrt{b}\left(e^{2Dt} - D_1\right)}$$

for some constant $D_1 > 0$. Then we see that the function u_0 blows up as t approaches a finite time. Thus, the metric g is incomplete.

This result shows that the metric $g_{\mathbb{N}}$ is non-flat. Then by Theorem 2.2 and [12, Theorem 1], (M, g) is locally conformally flat. According to [19, Theorem 2], (M, g, f) is either the Gaussian soliton or isometric to the Bryant soliton. □

Finally, we observe that there is a gap in the dimension.

Corollary 2.1. *Let* (M^n, g, f), *with* $n \neq 5$, *be an irreducible non-trivial GRS and* *let* $d := \dim \mathfrak{iso}(M,g)$. *If* $d < \frac{1}{2}(n-1)n$ *then* $d \leq \frac{1}{2}(n-1)(n-2)+1$.

Proof of Corollary 2.1. The proof is by contradiction. Suppose that $d > \frac{1}{2}(n-1)(n-2)+1$. The group of isometries on (M,g) generates a Lie algebra of complete Killing vector fields, which is a sub-algebra of $\mathfrak{iso}(M,g)$. From the proofs of Theorem 2.1 and 2.2, there is an injective map from $\mathfrak{iso}(M,g)$ to that of a co-dimension one regular submanifold (M_c, g_c). Furthermore, the completeness of a vector field is preserved under the map. Thus, for each regular connected component,

$$\dim(\mathrm{Iso}(M_c, g_c)) > \frac{1}{2}(n-1)(n-2)+1.$$

By [53, Theorem 3.2], $\dim(\mathrm{Iso}(M_c, g_c)) = \frac{1}{2}n(n-1)$ and each (M_c, g_c) is a space form which is homogeneous and complete. Thus, by continuity, we go back to the case of Theorem 2.2 and $d = \frac{1}{2}n(n-1)$, a contradiction. □

2.3 Appendix

In this Appendix, we consider the case of each level set of a GRS is Euclidean, which was mentioned in the proof of Theorem 2.1. We first adapt the gradient Ricci soliton equation (2.2) to the cohomogeneity one setting, essentially using the methodology and notation of [30].

Let G be a Lie group acting isometrically on a Riemannian manifold (M,g). The action is of *cohomogeneity one* if the orbit space M/G is one-dimensional. In this case, we choose a unit speed geodesic $\gamma(t)$ that intersects all principal orbits perpendicularly. Then, it is possible to define a G-equivariant diffeomorphism $\Phi : I \times P \mapsto M_0$ given by

$$\Phi(t, hK) = h \cdot \gamma(t).$$

Here, $M_0 \subset M$ is an open dense subset, I is an open interval; $P = G/K$ where K is the istropy group along $\gamma(t)$. Then, the pullback metric is of the form

$$\Phi^*(g) = dt^2 + g_t$$

where g_t is a one-parameter family of G-invariant metrics on P. We let L denote the shape operator $L(X) = \nabla_X N$, where $N = \Phi_*(\partial_t)$ is a unit normal vector field. We will consider $L_t = L_{|\Phi(t\times P)}$ to be a one-parameter family of endomorphisms on TP via identification $T(\Phi(t \times P)) = TP$. Following [30], we have $\partial_t g = 2g_t \circ L_t$, that is for $X, Y \in TP$,

$$(\partial_t g)(X, Y) = 2g_t(L_t(X), Y).$$

From Gauss, Codazzi, and Riccati equations, we find that the Ricci curvature of (M_0, g) is totally determined by the geometry of the shape operator and how it evolves. Moreover, if the function f is invariant by the group action, then the gradient Ricci soliton equation (2.2) is reduced to

$$\begin{aligned} 0 &= -(\delta L) - \nabla \operatorname{tr} L, \\ \lambda &= -\operatorname{trace}(L') - \operatorname{trace}(L^2) + f'', \\ \lambda g_t(X, Y) &= \operatorname{Ric}_t(X, Y) - (\operatorname{trace} L) g_t(L(X), Y) \\ &\quad - g_t(L'(X), Y) + f' g_t(L(X), Y). \end{aligned} \tag{2.17}$$

where Ric_t denotes the Ricci curvature of (P, g_t), $\delta L = \sum_i \nabla_{e_i} L(e_i)$ for an orthonormal basis and $\operatorname{tr} T = \operatorname{tr}_{g_t} T_t$.

Now, we consider a GRS (M^n, g, f) $(n \geq 3)$ with local Euclidean level sets:

$$g := dt^2 + g_t = dt^2 + \sum_i h_i^2(t) dx_i^2, \tag{2.18}$$

where each function h_i is smooth. Observe that

$$2g_t \circ L_t = \partial_t g_t = 2 \sum_i \frac{h_i'}{h_i} h_i^2 dx_i^2.$$

From this, we get

$$L_t(\partial_i) = \frac{h_i'}{h_i} \partial_i, \quad L_t^2(\partial_i) = \left(\frac{h_i'}{h_i}\right)^2 \partial_i,$$

and

$$L_i'(\partial_i) = \left(\frac{h_i''}{h_i} - \left(\frac{h_i'}{h_i}\right)^2\right) \partial_i.$$

Consequently,

$$\begin{aligned} \operatorname{trace} L_t &= \sum_i \frac{h_i'}{h_i}, \quad \operatorname{trace} L_t^2 = \sum_i \left(\frac{h_i'}{h_i}\right)^2, \\ \operatorname{trace} L_t' &= \sum_i \left(\frac{h_i''}{h_i} - \left(\frac{h_i'}{h_i}\right)^2\right). \end{aligned}$$

Since the shape operator L satisfies the Riccati equation [40, page 117], the sectional curvature of the 2-plane section spanned by $e_i = \frac{\partial_i}{h_i}$ and N is given by

$$\begin{aligned} K(e_i, N) &= g \circ \left(-L' - L^2\right)(e_i, e_i) \\ &= -g\left(L'(e_i), e_i\right) - g\left(L^2(e_i), e_i\right) \\ &= -g\left(\left(\frac{h_i''}{h_i} - \left(\frac{h_i'}{h_i}\right)^2\right) e_i, e_i\right) - g\left(\left(\frac{h_i'}{h_i}\right)^2 e_i, e_i\right) = -\frac{h_i''}{h_i}. \end{aligned}$$

Using the Gauss equation [83, Theorem 3.2.4], we see that the sectional curvature of the 2-plane section spanned by e_i and e_j is given by

$$K(e_i, e_j) = -g\left(L(e_i), e_i\right) g\left(L(e_j), e_j\right) = -\frac{h_i'}{h_i}\frac{h_j'}{h_j}.$$

The Ricci curvature is then given by

$$\operatorname{Ric}(N, N) = \sum_i K(e_i, N) = -\sum_i \frac{h_i''}{h_i},$$

and

$$\begin{aligned} \operatorname{Ric}(e_j, e_j) &= K(e_j, N) + \sum_{i \neq j} K(e_j, e_i) \\ &= -\frac{h_j''}{h_j} - \left(\sum_{i \neq j} \frac{h_i'}{h_i}\right)\frac{h_j'}{h_j} = -\frac{h_j''}{h_j} - \left(\sum_i \frac{h_i'}{h_i} - \frac{h_j'}{h_j}\right)\frac{h_j'}{h_j} \\ &= -\left(\sum_i \frac{h_i'}{h_i}\right)\frac{h_j'}{h_j} - \left(\frac{h_j''}{h_j} - \left(\frac{h_j'}{h_j}\right)^2\right). \end{aligned}$$

From these results, we imply that the scalar curvature is given by

$$\begin{aligned} S &= \operatorname{Ric}(N, N) + \sum_j \operatorname{Ric}(e_j, e_j) \\ &= -\sum_i \frac{h_i''}{h_i} - \left(\sum_i \frac{h_i'}{h_i}\right)\left(\sum_j \frac{h_j'}{h_j}\right) - \sum_j \left(\frac{h_j''}{h_j} - \left(\frac{h_j'}{h_j}\right)^2\right) \\ &= -2\sum_i \frac{h_i''}{h_i} - A^2 + B, \end{aligned}$$

where

$$A := \sum_i \frac{h_i'}{h_i}, B := \sum_i \left(\frac{h_i'}{h_i}\right)^2.$$

Thus, generically, the Weyl tensor is NOT vanishing.

Plugging the above results in (2.17), we conclude that

$$\lambda = -\sum_i \left(\frac{h_i''}{h_i} - \left(\frac{h_i'}{h_i} \right)^2 \right) - \sum_i \left(\frac{h_i'}{h_i} \right)^2 + f''$$
$$= \left(f' - \sum_i \frac{h_i'}{h_i} \right) \frac{h_j'}{h_j} - \left(\frac{h_j''}{h_j} - \left(\frac{h_j'}{h_j} \right)^2 \right).$$

Let $u_0 := f'$ and $u_i := \frac{h_i'}{h_i}$, the above system can be written as follows

$$\begin{cases} A = \Sigma_i u_i \\ B = \Sigma_i u_i^2 \\ u_j' = (u_0 - A)\, u_j - \lambda \\ u_0' = B + (u_0 - A)\, A - (n-1)\lambda. \end{cases} \tag{2.19}$$

This is a system of first-order ODEs, and the Picard-Lindelöf theorem yields local existence and uniqueness.

Chapter 3

Liouville type theorems and gradient estimates for nonlinear heat equations along ancient k-super Ricci flow via reduced geometry

Recall that for each real number k, a time-dependent Riemannian manifold $(M, g(x,t))_{t\in I}$ is called a k-super Ricci flow if it satisfies the following condition

$$\frac{\partial g}{\partial t} + 2\,\mathrm{Ric} \geq 2kg. \tag{3.1}$$

A k-super Ricci flow $(M, g(x,t))_{t\in I}$ is said to be ancient when $I = (-\infty, 0]$. Written based on the paper "Ha Tuan Dung, Nguyen Tien Manh, and Nguyen Dang Tuyen (2023), Liouville type theorems and gradient estimates for nonlinear heat equations along ancient K-super Ricci flow via reduced geometry, *Journal of Mathematical Analysis and Applications*, Vol. 519 (2), 126836" [32], Chapter 2 delves into the study of Liouville type theorems and gradient estimates for the positive bounded solutions to the nonlinear parabolic equation concerning Perelman's reduced distance

$$\frac{\partial}{\partial t}u(x,t) = \Delta u(x,t) + au(x,t)\ln u(x,t) \tag{3.2}$$

along ancient k-super Ricci flow $(M, g(x,t))_{t\in(-\infty,0]}$, where a is a real number. This is the content of Problem 1.2 that was discussed in Chapter 1. As in [56], we also work on the reverse time parameter $\tau := -t$. On this parameter, the ancient K-super Ricci flow $(M, g(t))_{t\in(-\infty,0]}$ becomes backward k-super Ricci flow $(M, g(\tau))_{\tau\in[0,\infty)}$, namely,

$$\mathrm{Ric} \geq \frac{1}{2}\frac{\partial g}{\partial \tau} + kg.$$

Moreover, the equation (3.2) can be translated as follows

$$\left(\frac{\partial}{\partial \tau} + \Delta\right) u(x,t) = -au(x,t)\ln u(x,t). \tag{3.3}$$

The chapter basically consists of two parts as follows.

In the first part of the chapter, we focus on recalling fundamental results in reduced geometry, along with several related problems that serve as the foundation for proving our main findings. Additionally, the chapter's primary results are thoroughly presented in this section.

In the next part of the chapter, we aim to formulate and prove Hamilton type gradient estimates for positive smooth solutions u via the localization technique of Li-Yau to the nonlinear parabolic equation (3.3). These gradient estimates play a crucial role in establishing the Liouville type results, which will be presented in the final part of this section.

3.1 Preliminaries and main results

3.1.1 The reduced distance function of Perelman

In this section, we mainly recall some basic results of reduced geometry and some related problems, which will be used to prove our result. The main references of Section 3.1 are [27, 56, 112]. Throughout this section, we assume that $(M, g(x,\tau))_{\tau\in[0,\infty)}$ is an n-dimensional, complete time-dependent Riemannian manifold. Besides, we sometimes write $u(x,\tau)$ as u, and also write $\frac{\partial u}{\partial \tau}$ as $\partial_\tau u$ or u_τ. We begin by providing the definition of reduced distance.

Definition 3.1. *The $\mathcal{L}$-length of a curve $\gamma : [\tau_1, \tau_2] \to M$ is defined as*

$$\mathcal{L}(\gamma) := \int_{\tau_1}^{\tau_2} \sqrt{\tau}\left(H + \left|\frac{d\gamma}{d\tau}\right|^2\right) d\tau,$$

where

$$\mathfrak{h} := \frac{1}{2}\partial_\tau g, \quad H := \mathrm{tr}\,\mathfrak{h}.$$

Definition 3.2. *For each $(x,\tau) \in M \times (0,\infty)$, we define the L-distance $L(x,\tau)$ and the reduced distance $\rho(x,\tau)$ from a space-time base point $(x_0, 0)$ as follows*

$$L(x,\tau) := \inf_{\gamma} \mathcal{L}(\gamma), \quad \rho(x,\tau) := \frac{1}{2\sqrt{\tau}} L(x,\tau), \tag{3.4}$$

where we take the infimum over all curves $\gamma : [0,\tau] \to M$ with $\gamma(0) = x_0$ and $\gamma(\tau) = x$. If a curve attains the infimum of (3.4) then it is called minimal $\mathcal{L}$-geodesic from $(x_0, 0)$ to (x,τ).

Remark 3.1. In the static case $\partial_\tau g = 0$, we have $\rho(x,\tau) = \frac{d(x)^2}{4\tau}$, where $d(x)$ is the Riemannian distance from x_0 induced from g.

Definition 3.3. *Let $(M, g(x,\tau))_{\tau\in[0,\infty)}$ be a complete, time-dependent Riemannian manifold. If for each $\tau > 0$ there is $c^\tau \geq 0$ depending only on τ such that $h \geq -c^\tau g$ on $[0,\tau]$ then $(M, g(x,\tau))_{\tau\in[0,\infty)}$ is admissible.*

Remark 3.2. From the results of Ye (see Propositions 2.12, 2.13 in [112]), we see that the functions $L(\cdot,\tau)$ and $L(x,\cdot)$ are locally Lipschitz in $(M, g(\tau))$ and $(0,\infty)$, respectively when $(M, g(x,\tau))_{\tau\in[0,\infty)}$ is admissible. Moreover, they are differentiable almost everywhere. Besides, the admissibility also implies the existence of minimal $\mathcal{L}$-geodesic (see Proposition 2.8 in [112]).

Note that if $H \geq 0$ then by Definition 3.1, we deduce that $\mathcal{L}$ is non-negative, so is $\rho(x,\tau)$. From this observation, for $(x,\tau) \in M \times (0,\infty)$ and $H \geq 0$, we can define

$$\overline{L}(x,\tau) := 4\tau\rho(x,\tau) = \mathfrak{d}(x,\tau)^2.$$

Next, we list here the following helpful lemma whose proof is exactly the same as in the proof of (7.88), (7.89), and (7.90) in [27].

Lemma 3.1. [27, Lemma 7.44][114, Subsection 2.3] *Suppose that ρ is smooth at $(\overline{x},\overline{\tau}) \in M \times (0,\infty)$. Then we have*

$$\partial_\tau \rho = H - \frac{\rho}{\overline{\tau}} + \frac{1}{2\overline{\tau}^{\frac{3}{2}}} \mathcal{K}_{\mathcal{H}}, \tag{3.5}$$

$$|\nabla\rho|^2 = -H + \frac{\rho}{\overline{\tau}} - \frac{1}{\overline{\tau}^{\frac{3}{2}}} \mathcal{K}_{\mathcal{H}}, \tag{3.6}$$

$$\Delta\rho \leq -H + \frac{n}{2\overline{\tau}} - \frac{1}{2\overline{\tau}^{\frac{3}{2}}} \mathcal{K}_{\mathcal{H}} - \frac{1}{2\overline{\tau}^{\frac{3}{2}}} \mathcal{K}_{\mathcal{D}}, \tag{3.7}$$

at $(\overline{x},\overline{\tau})$*, where*

$$\mathcal{K}_{\mathcal{H}} := \int_0^{\overline{\tau}} \tau^{\frac{3}{2}} \mathcal{H}(\overline{X}) d\tau, \quad \mathcal{K}_{\mathcal{D}} := \int_0^{\overline{\tau}} \tau^{\frac{3}{2}} \mathcal{D}(\overline{X}) d\tau.$$

Remark 3.3. We may conclude that even if ρ is not smooth at $(\overline{x},\overline{\tau})$, the above inequalities hold in the barrier sense by employing the same barrier function as in the proof of Lemma 5.3 in [74].

To establish main results, we will use the following Müller quantity $\mathcal{D}(X)$ (see Definition 1.3 in [70]) and trace Harnack quantity $\mathcal{H}(X)$ (see Definition 1.5 in [70]):

$$\begin{aligned} \mathcal{D}(X) := \partial_\tau H - \Delta H - 2|\mathfrak{h}|^2 + 4 \operatorname{div} \mathfrak{h}(X) \\ - 2g(\nabla H, X) + 2\mathrm{Ric}(X,X) - 2\mathfrak{h}(X,X), \end{aligned} \tag{3.8}$$

$$\mathcal{H}(X) := -\partial_\tau H - \frac{H}{\tau} - 2g(\nabla H, X) + 2\mathfrak{h}(X,X), \tag{3.9}$$

where X is a (time-dependent) vector field.

Remark 3.4. For the convenience of the proof later, we divide $\mathcal{D}(X)$ into two parts: $\mathcal{D}(X) = \mathcal{D}_0(X) + 2\mathcal{R}(X)$, where

$$\mathcal{D}_0(X) := -\partial_\tau H - \Delta H - 2|\mathfrak{h}|^2 + 4 \operatorname{div} h(X) - 2g(\nabla H, X),$$

$$\mathcal{R}(X) := \mathrm{Ric}(X,X) - \mathfrak{h}(X,X).$$

We notice that if $(M, g(x,\tau))_{\tau\in[0,\infty)}$ is a backward k-super Ricci flow then

$$\mathcal{R}(X) = \mathrm{Ric}(X,X) - \frac{1}{2}\partial_\tau g(X,X) \geq kg(X,X) = k|X|^2. \tag{3.10}$$

The next lemma concerning the L-distance and the function $\mathfrak{d}$ plays a key role in the proof of Theorem 3.1.

Lemma 3.2. [56, Lemma 3.5 and 3.6] *Let* $k \geq 0$*. We assume that the reduced distance* ρ *is smooth at* $(\overline{x},\overline{\tau}) \in M \times (0,\infty)$ *and*

$$\mathcal{D}(X) \geq -2k\left(H + |X|^2\right), \quad \mathcal{H}(X) \geq -\frac{H}{\tau}, \quad H \geq 0,$$

for all vector fields X*. Then at* $(\overline{x},\overline{\tau})$ *we have the following estimates*

$$(\Delta + \partial_\tau)\,\overline{L} \leq 2n + 2k\overline{L} \quad \text{and} \quad |\nabla \mathfrak{d}|^2 \leq 3.$$

In order to state the results, we introduce some notations. For $R, T > 0$, let $\mathcal{Q}_{R,T}$ be

$$\mathcal{Q}_{R,T} := \{(x,\tau) \in M \times (0,T] \mid \mathfrak{d}(x,\tau) \leq R\}.$$

Throughout the next sections, we make use of the following notation

$$q^+ := \max\{q, 0\}, \quad q^- := \min\{q, 0\}.$$

3.1.2 Main results

The main purpose of this chapter is to extend and improve the results of Kunikawa-Sakurai [56] and Dung-Dung [31]. Our first main result is the following Hamilton type gradient estimate:

Theorem 3.1. *For $k \geq 0$, let $(M, g(x,\tau))_{\tau\in[0,\infty)}$ be an n-dimensional, admissible, complete backward $(-k)$-super Ricci flow. We assume*

$$\mathcal{D}(X) \geq -2k\left(H + |X|^2\right), \quad \mathcal{H}(X) \geq -\frac{H}{\tau}, \quad H \geq 0,$$

for all vector fields X. Let $u : M \times [0,\infty) \to (0,\infty)$ be a positive solution to backward nonlinear heat equation (3.3). For $R, T > 0$ and $B > 0$, we suppose $u \leq B$ in the cylinder $\mathcal{Q}_{R,T}$. Then there exists a positive constant $c = c(n)$ depending only on n such that

$$\frac{|\nabla u|}{u} \leq c\left(\frac{\sqrt{A}}{R} + \frac{1}{\sqrt{T}} + \sqrt{k} + \sqrt{\sup_{\mathcal{Q}_{R,T}}\left\{[a(2+2\ln B - \ln u)]^+\right\}}\right)\sqrt{1+\ln\frac{B}{u}} \tag{3.11}$$

in $\mathcal{Q}_{\frac{R}{2},\frac{T}{4}}$, where $A = 1 + \ln B - \ln\left(\inf_{\mathcal{Q}_{R,T}} u\right)$.

Remark 3.5. Theorem 3.1 can be regarded as a generalization along the backward $(-k)$-super Ricci flow of Theorem 1.1 in [31].

When $a = 0$, we can derive the following local space-only gradient estimate for the backward heat equation under the $(-k)$-super Ricci flow.

Corollary 3.1. *For $k \geq 0$, let $(M, g(x,\tau))_{\tau\in[0,\infty)}$ be an n-dimensional, admissible, complete backward $(-k)$-super Ricci flow. We assume*

$$\mathcal{D}(X) \geq -2k\left(H + |X|^2\right), \quad \mathcal{H}(X) \geq -\frac{H}{\tau}, \quad H \geq 0,$$

for all vector fields X. Let $u : M \times [0, \infty) \to (0, \infty)$ stands for a positive solution to the backward heat equation

$$\left(\frac{\partial}{\partial \tau} + \Delta\right) u = 0. \tag{3.12}$$

For $R, T > 0$ and $B > 0$, we suppose $u \leq B$ in the cylinder $\mathcal{Q}_{R,T}$. Then there exists a positive constant $c = c(n)$ depending only on n such that

$$\frac{|\nabla u|}{u} \leq c\left(\frac{\sqrt{A}}{R} + \frac{1}{\sqrt{T}} + \sqrt{k}\right)\sqrt{1 + \ln\frac{B}{u}}, \tag{3.13}$$

in $\mathcal{Q}_{\frac{R}{2},\frac{T}{4}}$, where $A = 1 + \ln B - \ln\left(\inf_{\mathcal{Q}_{R,T}} u\right)$.

Remark 3.6. Since $(\partial_\tau + \Delta) u = 0$, let $v = u + 1$; then v satisfies $(\partial_\tau + \Delta) v = 0$. Thus, without loss of generality, we may assume that $u \geq 1$. Then, we get $A = 1 + \ln B$ and the inequality (3.13) becomes

$$\frac{|\nabla u|}{u} \leq c\left(\frac{\sqrt{1 + \ln B}}{R} + \frac{1}{\sqrt{T}} + \sqrt{k}\right)\sqrt{1 + \ln\frac{B}{u}}.$$

Notice that $\sqrt{1 + \ln\frac{B}{u}} \leq 1 + \ln\frac{B}{u}$. Thus, our result can be seen as a significant improvement to Theorem 2.8 of Kunikawa-Sakurai [56].

As an application of Theorem 3.1, we have the following Liouville theorem for the backward nonlinear heat equation (3.3).

Theorem 3.2. *Let $(M, g(x,\tau))_{\tau\in[0,\infty)}$ be an n-dimensional, admissible, complete backward super Ricci flow. We assume*

$$\mathcal{D}(X) \geq 0, \quad \mathcal{H}(X) \geq -\frac{H}{\tau}, \quad H \geq 0 \tag{3.14}$$

for all vector fields X.

1. *When $a < 0$, let $u : M \times [0, \infty) \to (0, \infty)$ be a positive solution to backward nonlinear heat equation (3.3). If $e^{-2} \leq u \leq B$ for some constant $B < 1$, then u does not exist; if $e^{-2} \leq u \leq B$ for some constant $B \geq 1$, then $u \equiv 1$.*

2. *When $a = 0$:*

 2a. *If $u : M \times [0, \infty) \to (0, \infty)$ be a positive solution to backward heat* *equation* (3.12) *such that*

$$u(x,\tau) = \exp\left[o\left(\mathfrak{d}(x,\tau) + \tau\right)\right] \tag{3.15}$$

near infinity, then u *is constant.*

2b. *If* $u : M \times [0,\infty) \to \mathbb{R}$ *be a solution to backward heat equation* (3.12) *such that*

$$u(x,\tau) = o\left(\mathfrak{d}(x,\tau) + \sqrt{\tau}\right) \tag{3.16}$$

near infinity, then u *is constant.*

Remark 3.7. The first part of Theorem 3.2 can be regarded as a generalization along the backward super Ricci flow of Theorem 1.3 (part ii) in [105] and Corollary 1.3 in [31]. When $a = 0$, the part 2a of Theorem 3.2 is better than Theorem 2.2 in [56]. In particular, in the static case of $\mathfrak{h} = 0$, the part 2a is reduced to Corollary 1.2 in [31].

3.2 Gradient estimates for (3.3) along the backward $(-k)$-supper Ricci flow and Liouville type results

In this section, inspired by the work of Kunikawa-Sakurai [56], we will study gradient estimates for positive solutions to the nonlinear parabolic equation (3.3) along the backward $(-k)$-supper Ricci flow $(M, g(\tau))_{\tau\in[0,\infty)}$. Recall the system that u and g solve

$$\begin{cases} (\partial_\tau + \Delta)\, u = -au\ln u, \\ \mathrm{Ric} \geq \mathfrak{h} - kg, \end{cases} \tag{3.17}$$

where $k \geq 0$ and $\mathfrak{h} := \frac{1}{2}\partial_\tau g$. Suppose that u is a positive solution to the backward nonlinear heat equation (3.3). We now introduce an auxiliary function

$$h = \sqrt{1 + \ln\frac{B}{u}} = \sqrt{\ln\frac{D}{u}} \geq 1$$

on $\mathcal{Q}_{R,T}$, where $D = Be$. Then, we have

$$u = De^{-h^2} \quad \text{and} \quad \ln u = \ln D - h^2.$$

This implies

$$u_\tau = -2Dh_\tau he^{-h^2}, \quad \nabla u = -2Dh\nabla he^{-h^2},$$

and

$$\Delta u = -2Dhe^{-h^2}\left[\Delta h + |\nabla h|^2\left(\frac{1}{h} - 2h\right)\right].$$

As a consequence, from (3.3), we get

$$-2Dh_\tau he^{-h^2} = 2Dhe^{-h^2}\left[\Delta h + |\nabla h|^2\left(\frac{1}{h} - 2h\right)\right] - aDe^{-h^2}\left(\ln D - h^2\right).$$

which is equivalent to

$$\left(\partial_\tau + \Delta\right)h = |\nabla h|^2\left(2h - \frac{1}{h}\right) + \frac{a}{2}\left(\frac{\ln D}{h} - h\right). \tag{3.18}$$

3.2.1 Basic lemmas

Using the equality (3.18), we have the following computational lemma, which will play a significant part in the proof of Theorem 3.1.

Lemma 3.3. *Let $(M, g(x,\tau))_{\tau\in[0,\infty)}$ be an n-dimensional, admissible, complete backward $(-k)$-super Ricci flow ($k \geq 0$) and u be a positive solution to the backward nonlinear heat equation (3.3). Suppose that $u \leq B$ for all $(x,t) \in \mathcal{Q}_{R,T}$ where $B > 0$. Denote $h = \sqrt{1 + \ln\frac{B}{u}}$ and $w = |\nabla h|^2$. Then on the cylinder $\mathcal{Q}_{R,T}$, we have*

$$\left(\Delta + \partial_\tau\right)w \geq 2\left(2h - \frac{1}{h}\right)\langle\nabla w, \nabla h\rangle + 2\left(2 + \frac{1}{h^2}\right)w^2 - \left(2k + \mathcal{P}\right)w, \tag{3.19}$$

where $\mathcal{P} = \sup\limits_{\mathcal{Q}_{R,T}}\left\{\left[a(2 + 2\ln B - \ln u)\right]^+\right\}.$

Proof of Lemma 3.3. We first proof the following identity

$$w_\tau = -\left(\partial_\tau g\right)\left(\nabla h, \nabla h\right) + 2\left\langle\nabla\left(h_\tau\right), \nabla h\right\rangle. \tag{3.20}$$

We will apply local coordinates to conveniently compute the above equation. For each $x \in M$, let $\{e_1, e_2, \ldots, e_n\}$ be a local orthonormal frame field and $\{\xi^1, \xi^2, \ldots, \xi^n\}$ be its coframe field. Here we adopt the notation that subscripts in i, j and k, with $1 \leq i, j, k \leq n$, mean covariant differentiations in the e_i, e_j and e_k directions respectively. Then, we have

$$\nabla h = \sum_{i,j=1}^{n} g^{ij}\nabla_i h e_j,$$

where $[g^{ij}]$ is the inverse of the matrix $[g_{ij}]$ and $g_{ij} = \langle e_i, e_j \rangle$. This implies that

$$\begin{aligned}
|\nabla h|^2 = \langle \nabla h, \nabla h \rangle &= \left\langle \sum_{i,j=1}^{n} g^{ij} \nabla_i h e_j, \sum_{k,l=1}^{n} g^{kl} \nabla_k h e_l \right\rangle \\
&= \sum_{i,j,k,l=1}^{n} g^{ij} g^{kl} \nabla_i h \nabla_k h \langle e_j, e_l \rangle \\
&= \sum_{i,k,l=1}^{n} \left(\sum_{j=1}^{n} g^{ij} g_{jl} \right) g^{kl} \nabla_i h \nabla_k h \\
&= \sum_{i,k,l=1}^{n} \delta_l^i g^{kl} \nabla_i h \nabla_k h = \sum_{i,k=1}^{n} g^{ki} \nabla_i h \nabla_k h.
\end{aligned}$$

Thus, we get

$$\begin{aligned}
w_\tau &= \partial_\tau \left(|\nabla h|^2 \right) = \partial_\tau \left(\sum_{i,j=1}^{n} g^{ij} \nabla_i h \nabla_j h \right) \\
&= \sum_{i,j=1}^{n} \left(\partial_\tau g^{ij} \right) \nabla_i h \nabla_j h + \sum_{i,j=1}^{n} g^{ij} \partial_\tau \left(\nabla_i h \right) \nabla_j h + \sum_{i,j=1}^{n} g^{ij} \nabla_i h \partial_\tau \left(\nabla_j h \right) \\
&= \sum_{i,j=1}^{n} \left(\partial_\tau g^{ij} \right) \nabla_i h \nabla_j h + 2 \sum_{i,j=1}^{n} g^{ij} \nabla_i \left(\partial_\tau h \right) \nabla_j h \\
&= \sum_{i,j=1}^{n} \left(\partial_\tau g^{ij} \right) \nabla_i h \nabla_j h + 2 \left\langle \nabla \left(h_\tau \right), \nabla h \right\rangle . \qquad (3.21)
\end{aligned}$$

Moreover, from the identity $\sum_{j=1}^{n} g^{ij} g_{jl} = \delta_l^i$, we obtain

$$0 = \partial_\tau \left(\delta_l^i \right) = \partial_\tau \left(\sum_{j=1}^{n} g^{ij} g_{jl} \right) = \sum_{j=1}^{n} \left(\partial_\tau g^{ij} \right) g_{jl} + \sum_{j=1}^{n} g^{ij} \left(\partial_\tau g_{jl} \right).$$

Consequently,

$$\sum_{j=1}^{n} \left(\partial_\tau g^{ij} \right) g_{jl} = - \sum_{i=1}^{n} g^{ij} \left(\partial_\tau g_{jl} \right).$$

From this, we deduce that

$$\begin{aligned}
- \sum_{l=1}^{n} \sum_{j=1}^{n} g^{ij} \left(\partial_\tau g_{jl} \right) g^{lk} &= \sum_{l=1}^{n} \sum_{j=1}^{n} \left(\partial_\tau g^{ij} \right) g_{jl} g^{lk} \\
&= \sum_{j=1}^{n} \left(\partial_\tau g^{ij} \right) \left(\sum_{l=1}^{n} g_{jl} g^{lk} \right) = \sum_{j=1}^{n} \left(\partial_\tau g^{ij} \right) \delta_j^k.
\end{aligned}$$

This shows that
$$\partial_\tau g^{ij} = -\sum_{k=1}^{n}\sum_{l=1}^{n} g^{ik}\left(\partial_\tau g_{kl}\right) g^{lj}.$$
Then, we find that
$$\sum_{i,j=1}^{n}\left(\partial_\tau g^{ij}\right)\nabla_i h\nabla_j h = -\sum_{i,j,k,l=1}^{n}\left(\partial_\tau g_{kl}\right) g^{ik}g^{lj}\nabla_i h\nabla_j h. \tag{3.22}$$
Note that $g=\sum_{i,j=1}^{n} g_{ij}\xi^i\otimes\xi^j$. Thus, we have
$$\begin{aligned}
\left(\partial_\tau g\right)\left(\nabla h,\nabla h\right) &= \left(\sum_{i,j=1}^{n}\left(\partial_\tau g_{ij}\right)\xi^i\otimes\xi^j\right)\left(\nabla h,\nabla h\right)\\
&= \left(\sum_{i,j=1}^{n}\left(\partial_\tau g_{ij}\right)\xi^i\otimes\xi^j\right)\left(\sum_{k,l=1}^{n} g^{kl}\nabla_k h e_l, \sum_{p,q=1}^{n} g^{pq}\nabla_p h e_q\right)\\
&= \sum_{i,j,k,l,p,q=1}^{n}\left(\partial_\tau g_{ij}\right) g^{kl}g^{pq}\nabla_k h\nabla_p h\xi^i\otimes\xi^j\left(e_l,e_q\right)\\
&= \sum_{i,j,k,l,p,q=1}^{n}\left(\partial_\tau g_{ij}\right) g^{kl}g^{pq}\nabla_k h\nabla_p h\xi^i\left(e_l\right)\otimes\xi^j\left(e_q\right)\\
&= \sum_{i,j,k,l,p,q=1}^{n}\left(\partial_\tau g_{ij}\right) g^{kl}g^{pq}\nabla_k h\nabla_p h\delta_l^i\delta_q^j\\
&= \sum_{i,j,k,p=1}^{n}\left(\partial_\tau g_{ij}\right) g^{ki}g^{pj}\nabla_k h\nabla_p h = \sum_{i,j,k,l=1}^{n}\left(\partial_\tau g_{kl}\right) g^{ik}g^{lj}\nabla_i h\nabla_j h.
\end{aligned}$$
This and (3.22) entail that
$$\sum_{i,j=1}^{n}\left(\partial_\tau g^{ij}\right)\nabla_i h\nabla_j h = -\left(\partial_\tau g\right)\left(\nabla h,\nabla h\right).$$
From the above identity and (3.21), we obtain the identity (3.20). Plugging (3.18) into (3.20), we conclude that
$$\begin{aligned}
w_\tau &= -\left(\partial_\tau g\right)\left(\nabla h,\nabla h\right) + 2\left\langle\nabla\left(h_\tau\right),\nabla h\right\rangle\\
&= -\left(\partial_\tau g\right)\left(\nabla h,\nabla h\right) - 2\langle\nabla\Delta h,\nabla h\rangle + 2|\nabla h|^2\left\langle\nabla\left(2h-\frac{1}{h}\right),\nabla h\right\rangle\\
&\quad + 2\left(2h-\frac{1}{h}\right)\left\langle\nabla\left(|\nabla h|^2\right),\nabla h\right\rangle + a\left\langle\nabla\left(\frac{\ln D}{h}-h\right),\nabla h\right\rangle.
\end{aligned}$$
Notice that
$$\left\langle\nabla\left(2h-\frac{1}{h}\right),\nabla h\right\rangle = \left(2+\frac{1}{h^2}\right)|\nabla h|^2,$$

and

$$\nabla \left(\frac{\ln D}{h} - h \right) = -\ln D \frac{\nabla h}{h^2} - \nabla h = -\left(\frac{\ln D}{h^2} + 1 \right) \nabla h.$$

Therefore, we obtain

$$\begin{aligned} w_\tau = & -(\partial_\tau g)(\nabla h, \nabla h) - 2\langle \nabla \Delta h, \nabla h \rangle + 2\left(2h - \frac{1}{h}\right)\langle \nabla\left(|\nabla h|^2\right), \nabla h\rangle \\ & + 2\left(2 + \frac{1}{h^2}\right)|\nabla h|^4 - a\left(\frac{\ln D}{h^2} + 1\right)|\nabla h|^2. \end{aligned} \tag{3.23}$$

Using the Bochner-Weitzenböck formula (see [104, Theorem 1.1]), we have

$$\frac{1}{2}\Delta|\nabla h|^2 = \left|\nabla^2 h\right|^2 + \mathrm{Ric}(\nabla h, \nabla h) + \langle \nabla \Delta h, \nabla h \rangle.$$

This and (3.23) entail that

$$\begin{aligned} w_\tau = & -(\partial_\tau g)(\nabla h, \nabla h) - \Delta|\nabla h|^2 + 2\mathrm{Ric}(\nabla h, \nabla h) + 2\left|\nabla^2 h\right|^2 \\ & + 2\left(2h - \frac{1}{h}\right)\langle \nabla w, \nabla h\rangle + 2\left(2 + \frac{1}{h^2}\right)w^2 + a\left(\frac{\ln D}{h^2} + 1\right)w \\ = & -\Delta|\nabla h|^2 + 2\left|\nabla^2 h\right|^2 + 2\left(2 + \frac{1}{h^2}\right)w^2 - a\left(\frac{\ln D}{h^2} + 1\right)w \\ & + 2\left(2h - \frac{1}{h}\right)\langle \nabla w, \nabla h\rangle + 2\left[\mathrm{Ric}(\nabla h, \nabla h) - \frac{1}{2}(\partial_\tau g)(\nabla h, \nabla h)\right] \\ = & -\Delta w + 2\left|\nabla^2 h\right|^2 + 2\left(2h - \frac{1}{h}\right)\langle \nabla w, \nabla h\rangle + 2\left(2 + \frac{1}{h^2}\right)w^2 \\ & - a\left(\frac{\ln D}{h^2} + 1\right)w + 2\mathcal{R}(\nabla h), \end{aligned}$$

or equivalently

$$\begin{aligned} \Delta w + w_\tau \geq & \, 2\left|\nabla^2 h\right|^2 + 2\left(2h - \frac{1}{h}\right)\langle \nabla w, \nabla h\rangle + 2\left(2 + \frac{1}{h^2}\right)w^2 \\ & - a\left(\frac{\ln D}{h^2} + 1\right)w + 2\mathcal{R}(\nabla h). \end{aligned} \tag{3.24}$$

Since $h \geq 1$, we get

$$\begin{aligned} a\left(\frac{\ln D}{h^2} + 1\right) = \frac{a}{h^2}\left(\ln D + h^2\right) & = \frac{a}{h^2}(2 + 2\ln B - \ln u) \\ & \leq \frac{1}{h^2}\max\{a(2 + 2\ln B - \ln u), 0\} \\ & \leq \sup_{\mathcal{Q}_{R,T}}\{\max\{a(2 + 2\ln B - \ln u), 0\}\} = \mathcal{P}. \end{aligned}$$

Moreover, we have $\mathcal{R}(\nabla h) \geq -kw$. These inequalities, combined with (3.24), yield that

$$\Delta w + w_\tau \geq 2\left(2h - \frac{1}{h}\right)\langle \nabla w, \nabla h\rangle + 2\left(2 + \frac{1}{h^2}\right) w^2 - (2k + \mathcal{P})\, w.$$

The proof is complete. □

To prove the main theorems in this section, we recall the following useful space-time cut-off function in [8, 56, 57, 61].

Lemma 3.4. *Given $R, T > 0$, there exists a smooth cut-off function $\overline{\psi}(r,t)$ supported in $[0,\infty)\times[0,\infty)$ satisfying following conditions*

(i) $0 \leq \overline{\psi}(r,t) \leq 1$ *in* $[0,\infty)\times[0,\infty)$.

(ii) *The equalities* $\overline{\psi}(r,t) = 1$ *and* $\frac{\partial \overline{\psi}}{\partial r}(r,t) = 0$ *hold in* $[0, \frac{R}{2}]\times[0,\frac{T}{4}]$ *and* $[0,\frac{R}{2}]\times[0,\infty)$, *respectively.*

(iii) *When* $0 < \epsilon \leq 1$, *there is a constant* C_ϵ *such that*

$$-\frac{C_\epsilon \overline{\psi}^{\epsilon}}{R} \leq \frac{\partial \overline{\psi}}{\partial r} \leq 0, \quad \text{and} \quad \left|\frac{\partial^2 \overline{\psi}}{\partial r^2}\right| \leq \frac{C_\epsilon \overline{\psi}^{\epsilon}}{R^2}.$$

(iv) $\overline{\psi}(r,t) = 0$ *on* $[R,\infty)\times[\frac{T}{2},\infty)$ *and* $\frac{|\partial_\tau \overline{\psi}|}{\overline{\psi}^{\frac{1}{2}}} \leq \frac{C}{T}$ *on* $[0,\infty)\times[0,T]$ *for some* $C > 0$.

Now, we take a cut-off function $\overline{\psi} : [0,\infty)\times[0,\infty) \to [0,1]$ satisfying all conditions in Lemma 3.4. Our main goal is to prove that inequality (3.11) in Theorem 3.1 holds at every point (x,τ) in $\mathcal{Q}_{\frac{R}{2},\frac{T}{4}}$. For this purpose, we introduce a smooth cut-off function $\psi : [0,\infty)\times[0,\infty) \to [0,1]$ by

$$\psi(x,\tau) := \overline{\psi}(\mathfrak{d}(x,\tau),\tau). \tag{3.25}$$

Using the cut-off function ψ, we have the following lemma.

Lemma 3.5. *Let $(M, g(x,\tau))_{\tau\in[0,\infty)}$ be an n-dimensional, admissible, complete backward $(-k)$-super Ricci flow ($k \geq 0$) and u be a positive solution to the backward nonlinear heat equation (3.3). Suppose that $u \leq B$ for all $(x,t) \in \mathcal{Q}_{R,T}$ where $B > 0$ and*

$$\mathcal{R}(X) \geq -k|X|^2, \quad \mathcal{D}(X) \geq -2k\left(H + |X|^2\right), \quad \mathcal{H}(X) \geq -\frac{H}{\tau}, \quad H \geq 0,$$

for all vector fields X. We define h and w as in Lemma 4.1. If

$$\Phi = (\Delta + \partial_\tau)(\psi w) - \frac{2\langle \nabla\psi, \nabla(\psi w)\rangle}{\psi} - 2\left(2h - \frac{1}{h}\right)\langle \nabla(\psi w), \nabla h\rangle,$$

then

$$\psi w^2 \leq c\left(\frac{A^2}{R^4} + \frac{1}{T^2} + k^2 + \mathcal{P}^2\right) + \frac{2h^2}{1+2h^2}\Phi.$$

at every point in $\mathcal{Q}_{R,T}$ such that the reduced distance is smooth, where c denotes a constant depending only on n whose value may change from line to line in the following.

Proof of Lemma 3.5. By direct computations, we see that

$$\begin{aligned}
\Phi &= (\Delta + \partial_\tau)(\psi w) - \frac{2\langle \nabla\psi, \nabla(\psi w)\rangle}{\psi} - 2\left(2h - \frac{1}{h}\right)\langle \nabla(\psi w), \nabla h\rangle \\
&= \Delta(\psi w) + \partial_\tau(\psi w) - 2\left(2h - \frac{1}{h}\right)\langle \psi\nabla w + w\nabla\psi, \nabla h\rangle \\
&\quad - \frac{2\langle \nabla\psi, \psi\nabla w + w\nabla\psi\rangle}{\psi} \\
&= \psi(\Delta w + w_\tau) + w(\Delta\psi + \psi_\tau) - 2\frac{|\nabla\psi|^2}{\psi}w - 2\psi\left(2h - \frac{1}{h}\right)\langle \nabla w, \nabla h\rangle \\
&\quad - 2w\left(2h - \frac{1}{h}\right)\langle \nabla\psi, \nabla h\rangle. \qquad (3.26)
\end{aligned}$$

Plugging (3.19) into (3.26), we get

$$\begin{aligned}
\Phi \geq\ & 2\left(2 + \frac{1}{h^2}\right)\psi w^2 - (2k + \mathcal{P})\,w - 2\frac{|\nabla\psi|^2}{\psi}w \\
& - 2w\left(2h - \frac{1}{h}\right)\langle \nabla\psi, \nabla h\rangle w + w(\Delta\psi + \psi_\tau),
\end{aligned}$$

or equivalently

$$\begin{aligned}
4\psi w^2 \leq & \frac{2h^2}{1+2h^2}(2k + \mathcal{P})\psi w + \frac{4h(1-2h^2)}{1+2h^2}\langle \nabla h, \nabla\psi\rangle w \\
& + \frac{4h^2}{1+2h^2}\frac{|\nabla\psi|^2}{\psi}w - \frac{2h^2}{1+2h^2}w(\Delta\psi + \psi_\tau) + \frac{2h^2}{1+2h^2}\Phi. \qquad (3.27)
\end{aligned}$$

Since $\overline{L} = \mathfrak{d}^2$, we get

$$-w\Delta\psi = -w\left[\psi_r(\Delta + \partial_\tau)\mathfrak{d} + \psi_{rr}|\nabla\mathfrak{d}|^2\right] = -w\left[\psi_r(\Delta + \partial_\tau)\left(\sqrt{\overline{L}}\right) + \psi_{rr}|\nabla\mathfrak{d}|^2\right].$$

Observe that

$$
\begin{aligned}
(\Delta+\partial_\tau)(\sqrt{\overline{L}}) &= \nabla\left(\frac{\nabla\overline{L}}{2\sqrt{\overline{L}}}\right)+\frac{1}{2\sqrt{\overline{L}}}\partial_\tau\overline{L}\\
&= \frac{\Delta\overline{L}}{2\sqrt{\overline{L}}}-\frac{|\nabla\overline{L}|^2}{4\overline{L}\sqrt{\overline{L}}}+\frac{1}{2\sqrt{\overline{L}}}\partial_\tau\overline{L}\\
&= \frac{\Delta\overline{L}}{2\mathfrak{d}}-\frac{|\nabla\overline{L}|^2}{4\mathfrak{d}^3}+\frac{1}{2\mathfrak{d}}\partial_\tau\overline{L}=\frac{1}{2\mathfrak{d}}(\Delta+\partial_\tau)\overline{L}-\frac{|\nabla\overline{L}|^2}{4\mathfrak{d}^3}.
\end{aligned}
$$

Thus, we have

$$
-w\Delta\psi=-\frac{w\psi_r}{2\mathfrak{d}}(\Delta+\partial_\tau)\overline{L}+w\psi_r\frac{|\nabla\overline{L}|^2}{4\mathfrak{d}^3}-w\psi_{rr}|\nabla\mathfrak{d}|^2.
$$

Note that $\psi_r\le 0$. Using this fact and the results of Lemma 3.2, we find that

$$
\begin{aligned}
-w\Delta\psi &= \frac{w|\psi_r|}{2\mathfrak{d}}(\Delta+\partial_\tau)\overline{L}-w|\psi_r|\frac{|\nabla\overline{L}|^2}{4\mathfrak{d}^3}-w\psi_{rr}|\nabla\mathfrak{d}|^2\\
&\le \frac{w|\psi_r|}{2\mathfrak{d}}(\Delta+\partial_\tau)\overline{L}+w|\psi_{rr}||\nabla\mathfrak{d}|^2\\
&\le \frac{w|\psi_r|}{2\mathfrak{d}}(2n+2k\overline{L})+3w|\psi_{rr}|\\
&\le \frac{2n}{R}w|\psi_r|+kRw|\psi_r|+3w|\psi_{rr}|.
\end{aligned}
$$

Combining (3.27) and the above estimate, we obtain

$$
\begin{aligned}
4\psi w^2 \le{}& \frac{2h^2}{1+2h^2}(2k+\mathcal{P})\psi w+\frac{4h(1-2h^2)}{1+2h^2}\langle\nabla h,\nabla\psi\rangle w+\frac{4h^2}{1+2h^2}\frac{|\nabla\psi|^2}{\psi}w\\
&+\frac{2h^2}{1+2h^2}\left(\frac{2n}{R}w|\psi_r|+KRw|\psi_r|+3w|\psi_{rr}|\right)\\
&-\frac{2h^2}{1+2h^2}w\psi_\tau+\frac{2h^2}{1+2h^2}\Phi.
\end{aligned}
$$

Since $0<\frac{2h^2}{1+2h^2}\le 1$, from the above inequality, we conclude that

$$
\begin{aligned}
4\psi w^2 \le{}& (2k+\mathcal{P})\psi w+\frac{4h|1-2h^2|}{1+2h^2}|\langle\nabla h,\nabla\psi\rangle|w+2\frac{|\nabla\psi|^2}{\psi}w\\
&+\left[\left(\frac{2n}{R}+kR\right)w|\psi_r|+3w|\psi_{rr}|\right]+w|\psi_\tau|+\frac{2h^2}{1+2h^2}\Phi. \qquad (3.28)
\end{aligned}
$$

Next, we will use the Young's inequality and Lemma 4.2 to estimate upper bounds for each term of the right-hand side (RHS) of (3.28).

For the first term on the RHS of (3.28), we have

$$\begin{aligned}(2k+\mathcal{P})\psi w &= \left(\psi^{\frac{1}{2}}w\right)\left[\psi^{\frac{1}{2}}(2k+\mathcal{P})\right] \\ &\le \frac{1}{2}\psi w^2 + c\psi(2k+\mathcal{P})^2 \le \frac{1}{2}\psi w^2 + cK^2 + c\mathcal{P}^2. \end{aligned} \tag{3.29}$$

For the second term on the RHS of (3.28), we have

$$\begin{aligned}\frac{4h\left|1-2h^2\right|}{1+2h^2}|\langle\nabla h,\nabla\psi\rangle|w &\le 4h\frac{\left|1-2h^2\right|}{1+2h^2}|\nabla\psi||\nabla h|w \\ &\le 4h|\nabla\psi|w^{\frac{3}{2}} = 4h|\nabla\psi|\psi^{\frac{-3}{4}}\left(\psi w^2\right)^{\frac{3}{4}} \\ &\le \frac{1}{2}\psi w^2 + ch^4\frac{|\nabla\psi|^4}{\psi^3} \le \frac{1}{2}\psi w^2 + \frac{cA^2}{R^4}, \end{aligned} \tag{3.30}$$

where $A = 1 + \ln B - \ln\left(\inf_{\mathcal{Q}_{R,T}} u\right)$.

For the third term on the RHS of (3.28), we have

$$\begin{aligned}\frac{2|\nabla\psi|^2}{\psi}w &= 2\left(|\nabla\psi|^2\psi^{-\frac{3}{2}}\right)\left(\psi^{\frac{1}{2}}w\right) \\ &\le \frac{1}{2}\psi w^2 + c\frac{|\nabla\psi|^4}{\psi^3} \le \frac{1}{2}\psi w^2 + \frac{c}{R^4}. \end{aligned} \tag{3.31}$$

For the fourth term on the RHS of (3.28), we have

$$\begin{aligned}&\left(\frac{2n}{R}+KR\right)w\left|\psi_r\right| + 3w\left|\psi_{rr}\right| \\ &= \left(\psi^{\frac{1}{2}}w\right)\left[\left(\frac{2n}{R}+KR\right)\frac{\left|\psi_r\right|}{\psi^{\frac{1}{2}}}\right] + 3\left(\psi^{\frac{1}{2}}w\right)\frac{\left|\psi_{rr}\right|}{\psi^{\frac{1}{2}}} \\ &\le \frac{1}{2}\psi w^2 + c\left(\frac{2n}{R}+KR\right)^2\frac{\left|\psi_r\right|^2}{\psi} + \frac{1}{2}\psi w^2 + c\frac{\left|\psi_{rr}\right|^2}{\psi} \\ &\le \psi w^2 + \frac{8cn^2}{R^2}\frac{\left|\psi_r\right|^2}{\psi} + cK^2R^2\frac{\left|\psi_r\right|^2}{\psi} + \frac{c}{R^4} \\ &\le \psi w^2 + \frac{c}{R^4} + cK^2. \end{aligned} \tag{3.32}$$

Finally, we estimate the last term on RHS of (3.28):

$$w\left|\psi_\tau\right| = \left(\psi^{\frac{1}{2}}w\right)\left(\frac{\left|\psi_\tau\right|}{\psi^{\frac{1}{2}}}\right) \le \frac{1}{2}\left(\psi^{\frac{1}{2}}w\right)^2 + c\left(\frac{\left|\psi_\tau\right|}{\psi^{\frac{1}{2}}}\right)^2 \le \frac{1}{2}\psi w^2 + \frac{c}{T^2}. \tag{3.33}$$

Substituting (3.29)-(3.33) into (3.28), we deduce that

$$\psi w^2 \le c\left(\frac{A^2}{R^4}+\frac{1}{T^2}+k^2+\mathcal{P}^2\right)+\frac{2h^2}{1+2h^2}\Phi.$$

The proof is complete. □

3.2.2 Gradient estimates and some special cases

We will apply Lemma 3.5 and the maximum principle to prove Theorem 3.1 by adapting arguments of [56].

Proof of Theorem 3.1. Define functions h, w and ψ as in Lemma 3.3 and (3.25), respectively. For $\theta > 0$, we consider a compact subset $\mathcal{Q}_{R,T,\theta}$ of $\mathcal{Q}_{R,T}$ by $Q_{R,T,\theta} := \{(x,\tau)\in Q_{R,T} \mid \tau\in[\theta,T]\}$. Next, we fix a small $\theta\in\left(0,\frac{T}{4}\right)$. Assume that the space-time maximum of ψw is reached at some point $(\overline{x},\overline{\tau})$ in $\mathcal{Q}_{R,T,\theta}$. We will prove Theorem 3.1 in the following two cases according to the smoothness of the distance function ρ at $(\overline{x},\overline{\tau})$.

Case 1: ρ is smooth at $(\overline{x},\overline{\tau})$. From Lemma 3.5 and the fact that $0\le\psi\le 1$, we get

$$(\psi w)^2 \le \psi w^2 \le c\left(\frac{A^2}{R^4}+\frac{1}{T^2}+K^2+\mathcal{P}^2\right)+\frac{2h^2}{1+2h^2}\Phi \tag{3.34}$$

at $(\overline{x},\overline{\tau})$, where Φ is defined as in Lemma 3.5. Note that for backward $(-k)$-supper Ricci flow $(M,g(\tau))_{\tau\in(0,\infty)}$, the assumption for $\mathcal{R}(X)$ in Lemma 3.5 is satisfied. Since $(\overline{x},\overline{\tau})$ is a maximum point, we have

$$\Delta(\psi w)\le 0,\quad \partial_\tau(\psi w)\le 0,\quad \nabla(\psi w)=0$$

at $(\overline{x},\overline{\tau})$. By the definition of Φ (see Lemma 3.5), we deduce that $\Phi(\overline{x},\overline{\tau})\le 0$. Therefore, the inequality (3.34) implies that

$$(\psi w)^2(x,\tau)\le(\psi w)^2(\overline{x},\overline{\tau})\le c\left(\frac{A^2}{R^4}+\frac{1}{T^2}+k^2+\mathcal{P}^2\right)$$

for all $(x,\tau)\in\mathcal{Q}_{R,T,\theta}$. This shows that

$$(\psi w)(x,\tau)\le c\left(\frac{A}{R^2}+\frac{1}{T}+k+\mathcal{P}\right)$$

for all $(x,\tau) \in \mathcal{Q}_{R,T,\theta}$.

Case 2: ρ is non-smooth at $(\overline{x},\overline{\tau})$. Then there is a sufficiently small $\delta > 0$, a small open neighborhood U of $\overline{x}$ in M, and a smooth upper barrier function $\widehat{\rho}$ of the reduced distance ρ on $U \times (\overline{\tau}-\delta, \overline{\tau}+\delta)$ such that $\widehat{\rho}$ satisfies (3.5), (3.6) and (3.7) in Lemma 3.1 at $(\overline{x},\overline{\tau})$. Moreover, we define

$$\widehat{\mathfrak{d}}(x,\tau) := \sqrt{4\tau\widehat{\rho}(x,\tau)}, \quad \widehat{\psi}(x,\tau) := \psi(\overline{\mathfrak{d}}(x,\tau),\tau),$$

where ψ is the function defined in Lemma 3.4. Note that $\widehat{\psi}$ is a smooth lower barrier of ψ at $(\overline{x},\overline{\tau})$. Besides, $(\overline{x},\overline{\tau})$ is the maximum point of $\widehat{\psi}w$ on $U \times (\overline{\tau}-\delta,\overline{\tau}+\delta) \cap Q_{R,T,\theta}$. Therefore, we conclude that

$$\Delta(\widehat{\psi}w) \leq 0, \quad \partial_\tau(\widehat{\psi}w) \leq 0, \quad \nabla(\widehat{\psi}w) = 0$$

at $(\overline{x},\overline{\tau})$. We can apply Lemma 3.2 combine with the above conditions of $\widehat{\psi}w$ to get

$$\begin{aligned}(\psi w)(x,\tau) &\leq (\psi w)(\overline{x},\overline{\tau}) \\ &= \left(\widehat{\psi}w\right)(\overline{x},\overline{\tau}) \\ &\leq \sqrt{\left(\widehat{\psi}w\right)^2(\overline{x},\overline{\tau})} \leq c\left(\frac{A}{R^2}+\frac{1}{T}+K+\mathcal{P}\right)\end{aligned}$$

for all $(x,\tau) \in \mathcal{Q}_{R,T,\theta}$.

In both cases, since $\psi \equiv 1$ on $\mathcal{Q}_{\frac{R}{2},\frac{T}{4}}$, by the definition of w and h, we have the following estimate

$$\frac{|\nabla u|}{u} \leq c\left(\frac{\sqrt{A}}{R}+\frac{1}{\sqrt{T}}+\sqrt{K}+\sqrt{\mathcal{P}}\right)\sqrt{1+\ln\frac{B}{u}},$$

on $\mathcal{Q}_{\frac{R}{2},\frac{T}{4},\theta}$. Thus, by letting $\theta \to 0$, the proof of Theorem 4.1 is complete. □

Remark 3.8. In the static case $\frac{\partial g}{\partial \tau} = 0$, we obtain

$$\mathfrak{h} = \tfrac{\partial g}{\partial \tau} = 0, \quad H = \operatorname{tr}\mathfrak{h} = 0, \quad \text{and} \quad \mathfrak{d}(x,t) = d(x).$$

Moreover, from the definition of $\mathcal{D}(X)$ and $\mathcal{H}(X)$, we imply that

$$\operatorname{Ric} \geq -kg, \quad \mathcal{H}(X) = -\frac{H}{\tau} = 0,$$

$$\mathcal{R}(X) = \operatorname{Ric}(X,X) \geq -kg\,(X,X) = -k|X|^2,$$

$$\mathcal{D}(X) = 2\,\mathrm{Ric}(X,X) \geq -2k|X|^2 = -2k\left(H + |X|^2\right),$$

for all vector fields X. Clearly, the conditions in Theorem 3.1 are satisfied in this case. Thus we can apply Theorem 3.1 to give local gradient estimates for positive solutions of the equation (3.2) on the static Riemannian manifold with $\mathrm{Ric} \geq -kg$.

Corollary 3.2. *Let (M,g) be an n-dimensional complete Riemannian manifold with $\mathrm{Ric} \geq -Kg$ for some constant $K \geq 0$ in*

$$B\left(x_0, R\right) = \left\{x \in M |\, d\left(x, x_0\right) = d\left(x\right) \leq R\right\}$$

for some fixed point x_0 in M, and some fixed radius $R > 0$. Let $u : M \times [0,\infty) \to (0,\infty)$ stand for a positive solution to the nonlinear heat equation (3.2). For $T > 0$ and $B > 0$, we suppose $u \leq B$ in the cylinder

$$\mathcal{Q}_{R,T} := B\left(x_0, R\right) \times (0,T] \subset M \times (0,T].$$

Then there exists a positive constant $c = c(n)$ depending only on n such that

$$\frac{|\nabla u|}{u} \leq c\left(\frac{\sqrt{A}}{R} + \frac{1}{\sqrt{T}} + \sqrt{K} + \sqrt{\sup_{\mathcal{Q}_{R,T}}\left\{[a(2 + 2\ln B - \ln u)]^+\right\}}\right)\sqrt{1 + \ln\frac{B}{u}}, \tag{3.35}$$

in $\mathcal{Q}_{\frac{R}{2},\frac{T}{4}}$, where $A = 1 + \ln B - \ln\left(\inf_{\mathcal{Q}_{R,T}} u\right)$.

Remark 3.9. Using the inequality $\ln(1+x) \leq x$ for all $x \geq 0$, we see that

$$\sqrt{1 + \ln\frac{B}{u}} \leq \sqrt{\frac{B}{u}}.$$

Then we can rewrite the inequality (3.35) in the case $1 \leq u \leq B$ as

$$\frac{|\nabla u|}{\sqrt{u}} \leq c\left(\frac{\sqrt{1 + \ln B}}{R} + \frac{1}{\sqrt{T}} + \sqrt{K} + \sqrt{\sup_{\mathcal{Q}_{R,T}}\left\{\max\left\{a(2 + 2\ln B - \ln u), 0\right\}\right\}}\right).$$

This shows that Corollary 3.2 is better than Theorem 1.3 of Jiang [50] and Theorem 1.1 of Wu [105] in the case $a < 0$.

Using Theorem 3.1, we can derive for positive solutions to the nonlinear parabolic equation (3.3) along complete backward Ricci flow with bounded, non-negative curvature operator.

Corollary 3.3. *For $k \geq 0$, let $(M, g(x,\tau))_{\tau\in[0,\infty)}$ be an n-dimensional, complete backward $(-k)$-Ricci flow with bounded, non-negative curvature operator. Let $u : M \times [0,\infty) \to (0,\infty)$ be a positive solution to backward nonlinear heat equation (3.3). For $R, T > 0$ and $B > 0$, we suppose $u \leq B$ in the cylinder $\mathcal{Q}_{R,T}$. Then there exists a positive constant $c = c(n)$ depending only on n such that*

$$\frac{|\nabla u|}{u} \leq c\left(\frac{\sqrt{A}}{R} + \frac{1}{\sqrt{T}} + \sqrt{k} + \sqrt{\sup_{\mathcal{Q}_{R,T}}\left\{[a(2 + 2\ln B - \ln u)]^+\right\}}\right)\sqrt{1 + \ln\frac{B}{u}} \tag{3.36}$$

in $\mathcal{Q}_{\frac{R}{2},\frac{T}{4}}$, where $A = 1 + \ln B - \ln\left(\inf_{\mathcal{Q}_{R,T}} u\right)$.

Proof of Corollary 3.3. By the assumption, we have $\mathfrak{h} = \mathrm{Ric} + kg$. Consequently,

$$H = \mathrm{tr}\,\mathfrak{h} = \mathrm{tr}\,(\mathrm{Ric} + kg) = \mathrm{tr}\,\mathrm{Ric} + \mathrm{tr}\,(kg) = S + nk,$$

for the scalar curvature S. In addition,

$$\mathcal{R}(X) = \mathrm{Ric}(X,X) - \frac{1}{2}\partial_\tau g(X,X) = -kg(X,X) = -k|X|^2,$$

for all vector fields X. Using the contracted second Bianchi identity, we obtain

$$\mathcal{D}_0(X) = -\partial_\tau S - \Delta S - 2|\mathrm{Ric}|^2 - 4kS - 2nk^2. \tag{3.37}$$

By Proposition 4.10 in [2], we have the following evolution formula for S

$$\partial_\tau S = -\Delta S - 2|\mathrm{Ric}|^2 - 2kS.$$

This and (3.37) entail that

$$\mathcal{D}_0(X) = -2kS - 2nk^2 = -2k\,(S + nk) = -2kH.$$

Thus, we have

$$\mathcal{D}(X) = \mathcal{D}_0(X) + 2\mathcal{R}(X) = -2kH - 2k|X|^2 = -2k\left(H + |X|^2\right). \tag{3.38}$$

From (3.9) and $\mathfrak{h} = \mathrm{Ric} + kg$, we get

$$\begin{aligned}\mathcal{H}(X) + \frac{H}{\tau} &= -\partial_\tau H - 2g(\nabla H, X) + 2\mathfrak{h}(X,X)\\ &= -\partial_\tau S - 2g(\nabla S, X) + 2\,\mathrm{Ric}(X,X) + 2k|X|^2 \geq 0,\end{aligned} \tag{3.39}$$

where we used Corollary 5.3 in [56] and the assumption $k \geq 0$. From the non-negativity of curvature operator, we see that the admissibility of $(M, g(x,\tau))_{\tau\in[0,\infty)}$ is satisfied and $H \geq 0$. Combining this with (3.38) and (3.39), we conclude that $(M, g(x,\tau))_{\tau\in[0,\infty)}$ satisfies the assumption in Theorem 3.1. We complete the proof of Corollary 3.3. □

3.2.3 Liouville type results

We now apply Theorem 3.1 to prove Theorem 3.2.

Proof of Theorem 3.2. Let $(M, g(x,\tau))_{\tau\in[0,\infty)}$ be backward super Ricci flow satisfying (3.14) for all vector fields X. By the assumption of Theorem 3.2, we have $k = 0$. Suppose that $u : M \times [0,\infty) \to (0,\infty)$ is a positive solution to the backward nonlinear heat equation (3.3). We fix $(x,\tau) \in \mathcal{Q}_{R,R} \subset M \times (0,\infty)$. For every sufficiently large $R > 0$, we see $(x,\tau) \in \mathcal{Q}_{\frac{R}{2},\frac{R}{4}}$.

1. When $a \leq 0$ and $e^{-2} \leq u \leq B$ for some constant B, we have $A \leq 3 + \ln B < \infty$ and

$$a(2 + 2\ln B - \ln u) = a\,(2 + \ln B) + a \ln \frac{B}{u} \leq 0.$$

This implies that

$$\sup_{\mathcal{Q}_{R,R}} \left\{ [a(2 + 2\ln B - \ln u)]^+ \right\} = 0.$$

From Theorem 3.1, we have

$$\frac{|\nabla u|}{u} \leq c \left(\frac{\sqrt{3 + \ln B}}{R} + \frac{1}{\sqrt{R}} \right) \sqrt{1 + \ln \frac{B}{u}}, \tag{3.40}$$

in $\mathcal{Q}_{R,R}$. Letting $R \to \infty$ in (3.40), we obtain $|\nabla u\,(x,\tau)| = 0$. This shows that u must be constant in x. Substituting $u(x,\tau) = u(\tau)$ into the equation (3.3), we get the following ordinary differential equation (ODE)

$$\partial_\tau u = -au \ln u.$$

If $a = 0$, then we can obtain $\partial_\tau u = 0$ and u is constant. If $a < 0$, solving the above ODE, we deduce that

$$u(\tau) = \exp\left\{ q e^{-a\tau} \right\}$$

for some $q \in \mathbb{R}$. When $\tau \to +\infty$, we see that: $u(\tau) = \exp\left\{ q e^{-a\tau} \right\} \to +\infty$

if $q > 0$; $u(\tau) = \exp\{qe^{-a\tau}\} \to 0$ if $q < 0$. From this and the assumption $e^{-2} \le u(x,\tau) \le B$, we see that u only exists when $B \ge 1$. The proof is complete.

2. When $a = 0$, $u : M \times [0,\infty) \to (0,\infty)$ is then a positive solution to backward heat equation (3.12). Notice that u and $v = u+1$ has the same growth at infinity. By Remark 3.6, we may assume that $u \ge 1$. Using Theorem 3.1, we get

$$\frac{|\nabla u|}{u} \le c\left(\frac{\sqrt{1+\ln B}}{R} + \frac{1}{\sqrt{R}}\right)\sqrt{1+\ln\frac{B}{u}}, \tag{3.41}$$

in $\mathcal{Q}_{R,R}$. For $R > 0$, we denote $A_R = \sup_{\mathcal{Q}_{R,R}} u$. The growth condition (3.15) yields

$$\ln(A_R) = o(R), \text{ as } R \to \infty.$$

Applying (3.41) to the function u on $\mathcal{Q}_{R,R}$, we have

$$\begin{aligned}\frac{|\nabla u|}{u} &\le c\left(\frac{\sqrt{1+\ln A_R}}{R} + \frac{1}{\sqrt{R}}\right)\sqrt{1+\ln A_R}\\ &= c\left(\frac{\sqrt{1+o(R)}}{R} + \frac{1}{\sqrt{R}}\right)\sqrt{1+\ln o(R)}\end{aligned}$$

at (x,τ). Letting $R \to \infty$ in the above inequality, we conclude that

$$|\nabla u(x,\tau)| = 0$$

and u is constant because (x,τ) is arbitrary. The proof of 2a is complete.

The proof of 2b is similar as in [56, 94], and we omit the details. □

Remark 3.10. When $(M, g(x,\tau))_{\tau\in[0,\infty)}$ is a complete backward Ricci flow with bounded, non-negative curvature operator, we obtain similar Liouville type results as in Theorem 3.2.

Chapter 4

Gradient estimates for a general type of nonlinear parabolic equations under geometric conditions and related problems

This chapter is written based on the paper "Ha Tuan Dung (2023), Gradient estimates for a general type of nonlinear parabolic equations under geometric conditions and related problems, *Nonlinear Analysis*, Vol. 226, 113135" [34]. In the present chapter, we establish gradient estimates for the positive bounded solutions to a general type of nonlinear parabolic equation concerning the weighted Laplacian

$$\left(\frac{\partial}{\partial t} - a(x,t) - \Delta_f\right) u(x,t) = F(u(x,t)) \tag{4.1}$$

on a smooth metric measure space with the metric evolving under the (k,∞)-super Perelman-Ricci flow (1.11) and the Yamabe flow (1.13), where $a(x,t)$ is a function which is $\mathcal{C}^2$ in the x-variable and $\mathcal{C}^1$ in the t-variable, and $F(u)$ is a $\mathcal{C}^2$ function of u. We derive several outcomes from these estimates, including Harnack inequalities, general global constancy, and Liouville type theorems. Applications related to some important geometric partial differential equations are presented to illustrate the strength of the results. The content of this chapter can be seen as a continuation of the work done previously in Chapter 3.

In order to state the main results in Chapter 4, we introduce some notations. On an n-dimensional smooth metric measure space $\left(M, g(x,t), e^{-f(x,t)}d\mu\right)_{t\in[0,T]}$ with the metric evolving under the geometric flow, we write $\operatorname{dist}(x, x_0, t)$ (or $r(x,t)$) for the Riemannian distance between $x \in M$ and x_0 with respect to the metric $g(x,t)$, where $x_0 \in M$ is a fixed point. We introduce the compact set

$$\mathcal{Q}_{R,T} := \{(x,t) \in M \times [0,T] \mid \operatorname{dist}(x, x_0, t) \leq R\},$$

where $R \geq 2$ and $T > 0$. Besides, we make use of the following notations

$$q^+ := \max\{q, 0\}, \quad q^- := \min\{q, 0\},$$

and

$$\mu = \max_{(x,t)} \{\Delta_f r(x,t) : \operatorname{dist}(x, x_0, t) = 1, 0 \leq t \leq T\}, \quad \mu^+ := \max\{\mu, 0\}.$$

On the static metric measure space $(M, g, e^{-f}d\mu)$, let $d(x, x_0)$ (or $r(x)$) denote the Riemannian distance to x from x_0 with respect to g, and $B(x_0, R)$ denote the geodesic ball centered at x_0 of radius $R \geq 2$. For $T > 0$, let $Q_{R,T}$ be

$$Q_{R,T} := B(x_0, R) \times [0,T] \subset M \times [0, \infty).$$

In this case, we also introduce the following quantities

$$\mu := \max_{\{x|d(x,x_0)=1\}} \Delta_f r(x), \quad \mu^+ := \max\{\mu, 0\}.$$

Our first main result states as follows.

Theorem 4.1. *Let $\left(M, g(x,t), e^{-f(x,t)}d\mu\right)_{t\in[0,T]}$ be a complete solution to the (k,∞)-super Perelman-Ricci flow (1.11) on an n-dimensional smooth manifold M and u be a smooth positive solution to the nonlinear heat equation (4.1) in $\mathcal{Q}_{R,T}$. Assume that $0 < u \leq B$ and*

$$\operatorname{Ric}_f \geq -(n-1)K, \quad \frac{\partial g}{\partial t} \geq -2Hg$$

for some $K, H \geq 0$ in $\mathcal{Q}_{R,T}$. Then there exists a constant c depending only n such that

$$\frac{|\nabla u|}{u} \leq c\left[\frac{\sqrt{A}}{R} + \sqrt{\frac{\mu^+}{R}} + \frac{1}{\sqrt{t}} + \sqrt[4]{(k^+)^2 + K^2 + H^2 + \mathcal{P}^2} + \Gamma_a\right]\sqrt{1 + \ln\frac{B}{u}} \tag{4.2}$$

for all $(x,t)\in \mathcal{Q}_{\frac{R}{2},T}$ *with* $t\neq 0$, *where* $A=1+\ln B-\ln\left(\inf_{\mathcal{Q}_{R,T}} u\right)$ *and*

$$\Gamma_a=\sup_{\mathcal{Q}_{R,T}}\left\{(a^+)^{\frac{1}{2}}+|\nabla a|^{\frac{1}{3}}\right\},$$

$$\mathcal{P}=\sup_{\mathcal{Q}_{R,T}}\left\{\left[2F'(u)-\frac{2F(u)}{u}+\frac{1}{1+\ln B-\ln u}\frac{F(u)}{u}\right]^+\right\}.$$

In the static case $\frac{\partial g}{\partial t}\equiv 0$ and $\frac{\partial f}{\partial t}\equiv 0$, we can set

$$H=0 \quad\text{and}\quad k=(n-1)K$$

in Theorem 4.1. Then, we see that $(M,g,e^{-f}d\mu)$ becomes a static smooth metric measure space where $\mathrm{Ric}_f\geq -(n-1)K$ for some constant $K\geq 0$ in the geodesic ball $B(x_0,R)$. From this observation and Theorem 4.1, we have the following result.

Theorem 4.2. *Let* $(M,g,e^{-f}d\mu)$ *be an* n*-dimensional complete smooth metric measure space with* $\mathrm{Ric}_f\geq -(n-1)K$ *for some constant* $K\geq 0$ *in* $B(x_0,R)$. *Assume that* $0<u(x,t)\leq B$ *for some constant* B, *is a smooth solution to the nonlinear heat equation* (4.1) *in* $Q_{R,T}$. *Then there exists a constant* c *depending only* n *such that*

$$\frac{|\nabla u|}{u}\leq c\left[\frac{\sqrt{A}}{R}+\sqrt{\frac{\mu^+}{R}}+\frac{1}{\sqrt{t}}+\sqrt{K}+\sqrt{\mathcal{P}}+\Gamma_a\right]\sqrt{1+\ln\frac{B}{u}}. \tag{4.3}$$

for all $(x,t)\in Q_{\frac{R}{2},T}$ *with* $t\neq 0$, *where* $A=1+\ln B-\ln\left(\inf_{Q_{R,T}} u\right)$ *and*

$$\Gamma_a=\sup_{Q_{R,T}}\left\{(a^+)^{\frac{1}{2}}+|\nabla a|^{\frac{1}{3}}\right\},$$

$$\mathcal{P}=\sup_{Q_{R,T}}\left\{\left[2F'(u)-\frac{2F(u)}{u}+\frac{1}{1+\ln B-\ln u}\frac{F(u)}{u}\right]^+\right\}.$$

On the other hand, we can give a local gradient estimate for the positive bounded solutions to the general type of nonlinear parabolic equation (4.1) under the Yamabe flow.

Theorem 4.3. *Let* $\left(M,g(x,t),e^{-f(x,t)}d\mu\right)_{t\in[0,T]}$ *be a complete solution to the Yamabe flow* (1.13) *on an* n*-dimensional smooth manifold* M *and* u *be a smooth positive solution to the nonlinear heat equation* (4.1) *in* $\mathcal{Q}_{R,T}$. *Assume that* $0<u\leq B$ *and* $\mathrm{Ric}_f\geq -(n-1)K, S\leq \mathcal{H}$ *for some* $K,\mathcal{H}\geq 0$ *in* $\mathcal{Q}_{R,T}$. *Then there exists*

a constant c depending only n such that

$$\frac{|\nabla u|}{u} \leq c\left[\frac{\sqrt{A}}{R}+\sqrt{\frac{\mu^+}{R}}+\frac{1}{\sqrt{t}}+\sqrt[4]{K^2+\mathcal{H}^2+\mathcal{P}^2}+\Gamma_a\right]\sqrt{1+\ln\frac{B}{u}} \tag{4.4}$$

for all $(x,t) \in \mathcal{Q}_{\frac{R}{2},T}$ with $t \neq 0$, where $A, \Gamma_a, \mathcal{P}$ are the same as Theorem 4.1.

Chapter 4 is organized as follows. In Section 4.1, we provide a proof of gradient estimates under the (k,∞)-super Perelman-Ricci flow and some corollaries. In Section 4.2, we study gradient estimates of (4.1) under the Yamabe flow and give a proof of Theorem 4.2. Gradient estimates and Liouville type results for some important geometric partial differential equations are given in Section 4.3.

4.1 Gradient estimates for (4.1) under the (k,∞)-super Perelman-Ricci flow

In this section, inspired by the work of Taheri [97], we will study gradient estimates for positive solutions of the general type of nonlinear parabolic equation (4.1) under the (k,∞)-super Perelman-Ricci flow $\left(M, g(x,t), e^{-f(x,t)}d\mu\right)_{t\in[0,T]}$. Recall the system that u and g solve

$$\begin{cases} u_t = \Delta_f u + au + F(u), \\ \dfrac{\partial g}{\partial t} + 2\mathrm{Ric}_f \geq -2kg, \end{cases}$$

with $k \in \mathbb{R}$. Here, Ric_f is the Bakry-Émery curvature. We now introduce an auxiliary function

$$h = \sqrt{1+\ln\frac{B}{u}} = \sqrt{\ln\frac{D}{u}} \geq 1$$

in $\mathcal{Q}_{R,T}$, where $D = Be$. Then, we have

$$u = De^{-h^2},\ F(u) = F\left(De^{-h^2}\right) \text{ and } \ln u = \ln D - h^2.$$

This implies

$$u_t = -2Dh_t he^{-h^2}, \quad \nabla u = -2Dh\nabla he^{-h^2},$$

and

$$\Delta_f u = \Delta u - \langle \nabla f, \nabla u\rangle = -2Dhe^{-h^2}\left[\Delta_f h + |\nabla h|^2\left(\frac{1}{h} - 2h\right)\right].$$

As a consequence, from (4.1), we get

$$-2Dh_the^{-h^2} = -2Dhe^{-h^2}\left[\Delta_f h + |\nabla h|^2\left(\frac{1}{h} - 2h\right)\right] + aDe^{-h^2} + F\left(De^{-h^2}\right),$$

which is equivalent to

$$h_t = \Delta_f h + |\nabla h|^2\left(\frac{1}{h} - 2h\right) - \frac{a}{2h} - \frac{F\left(De^{-h^2}\right)}{2Dhe^{-h^2}}. \tag{4.5}$$

Using the above equality, we have the following computational lemma, which will play an important part in the proof of Theorem 4.1.

Lemma 4.1. *Under the same assumption as in Theorem 4.1, for all (x,t) in $\mathcal{Q}_{R,T}$, the function $w = |\nabla h|^2$ satisfies*

$$\begin{aligned}\Delta_f w - w_t \geq 2\left(2h - \frac{1}{h}\right)\langle \nabla w, \nabla h\rangle - |\nabla a| w^{\frac{1}{2}} \\ - \left(2k^+ + a^+ + G\right) w + 2\left(2 + \frac{1}{h^2}\right) w^2,\end{aligned} \tag{4.6}$$

where

$$G = \left[2F'(u) - \frac{2F(u)}{u} + \frac{1}{1 + \ln B - \ln u}\frac{F(u)}{u}\right]^+.$$

Proof of Lemma 4.1. Using the same arguments as in proving equality (3.20), we get

$$w_t = -\frac{\partial g}{\partial t}(\nabla h, \nabla h) + 2\left\langle \nabla\left(h_t\right), \nabla h\right\rangle. \tag{4.7}$$

Now we recall the Bochner-Weitzenböck formula, according to which

$$\frac{1}{2}\Delta_f |\nabla h|^2 = \left|\nabla^2 h\right|^2 + \mathrm{Ric}_f(\nabla h, \nabla h) + \langle \nabla \Delta_f h, \nabla h\rangle.$$

This and (4.7) entail that

$$\begin{aligned}&\Delta_f w - w_t \\ &= \Delta_f |\nabla h|^2 + \frac{\partial g}{\partial t}(\nabla h, \nabla h) - 2\left\langle \nabla\left(h_t\right), \nabla h\right\rangle \\ &= 2\left|\nabla^2 h\right|^2 + \left[\frac{\partial g}{\partial t}(\nabla h, \nabla h) + 2\,\mathrm{Ric}_f(\nabla h, \nabla h)\right] + 2\left\langle \nabla\left(\Delta_f h - h_t\right), \nabla h\right\rangle.\end{aligned}$$

Our assumption on the (k, ∞)-super Perelman-Ricci flow of M and (4.5) imply the inequality

$$
\begin{aligned}
&\Delta_f w - w_t \\
\geq & -2kw + 2 + 2\left\langle \nabla\left(\Delta_f h - h_t\right), \nabla h\right\rangle \\
= & -2kw - 2\left\langle \nabla\left(h_t\right), \nabla h\right\rangle \\
& + 2\left\langle \nabla\left(h_t + |\nabla h|^2\left(2h - \frac{1}{h}\right) + \frac{a}{2h} + \frac{(De^{-h^2})^{\alpha-1}}{2h}\right), \nabla h\right\rangle \\
\geq & -2kw + 2\left(2h - \frac{1}{h}\right)\left\langle \nabla\left(|\nabla h|^2\right), \nabla h\right\rangle \\
& + 2|\nabla h|^2\left\langle \nabla\left(2h - \frac{1}{h}\right), \nabla h\right\rangle + \left\langle \nabla\left(\frac{a}{h}\right), \nabla h\right\rangle + \left\langle \nabla\left(\frac{F\left(De^{-h^2}\right)}{Dhe^{-h^2}}\right), \nabla h\right\rangle .
\end{aligned} \tag{4.8}
$$

On the other hand, we have

$$
\left\langle \nabla\left(2h - \frac{1}{h}\right), \nabla h\right\rangle = \left(2 + \frac{1}{h^2}\right)|\nabla h|^2, \tag{4.9}
$$

$$
\left\langle \nabla\left(\frac{a}{h}\right), \nabla h\right\rangle = \left\langle \frac{\nabla a}{h} - \frac{a\nabla h}{h^2}, \nabla h\right\rangle = \frac{1}{h}\left\langle \nabla a, \nabla h\right\rangle - \frac{a}{h^2}|\nabla h|^2, \tag{4.10}
$$

and

$$
\begin{aligned}
&\nabla\left(\frac{F\left(De^{-h^2}\right)}{Dhe^{-h^2}}\right) \\
&= \frac{\nabla\left(F\left(De^{-h^2}\right)\right)}{Dhe^{-h^2}} - \frac{F\left(De^{-h^2}\right)\nabla\left(Dhe^{-h^2}\right)}{D^2h^2e^{-2h^2}} \\
&= \frac{F'\left(De^{-h^2}\right)\nabla\left(De^{-h^2}\right)}{Dhe^{-h^2}} - \frac{F\left(De^{-h^2}\right)D\left[h\nabla\left(e^{-h^2}\right) + e^{-h^2}\nabla h\right]}{D^2h^2e^{-2h^2}} \\
&= \frac{-2Dhe^{-h^2}F'\left(De^{-h^2}\right)\nabla h}{Dhe^{-h^2}} - \frac{D\left(1 - 2h^2\right)e^{-h^2}F\left(De^{-h^2}\right)\nabla h}{D^2h^2e^{-2h^2}} \\
&= -2F'\left(De^{-h^2}\right)\nabla h - \left(\frac{1}{h^2} - 2\right)\frac{F\left(De^{-h^2}\right)}{De^{-h^2}}\nabla h \\
&= -\left[2F'\left(De^{-h^2}\right) - \frac{2F\left(De^{-h^2}\right)}{De^{-h^2}} + \frac{1}{h^2}\frac{F\left(De^{-h^2}\right)}{De^{-h^2}}\right]\nabla h.
\end{aligned}
$$

Consequently,

$$\begin{aligned}
&\left\langle \nabla \left(\frac{F\left(De^{-h^2}\right)}{Dhe^{-h^2}} \right), \nabla h \right\rangle \\
&= -\left[2F'\left(De^{-h^2}\right) - \frac{2F\left(De^{-h^2}\right)}{De^{-h^2}} + \frac{1}{h^2}\frac{F\left(De^{-h^2}\right)}{De^{-h^2}} \right] |\nabla h|^2.
\end{aligned} \tag{4.11}$$

Substituting (4.9)-(4.11) into (4.8), we deduce that

$$\begin{aligned}
\Delta_f w - w_t \geq & -2kw + 2\left(2h - \frac{1}{h}\right)\langle \nabla w, \nabla h\rangle + 2\left(2 + \frac{1}{h^2}\right)w^2 + \frac{1}{h}\langle \nabla a, \nabla h\rangle \\
& - \frac{a}{h^2}w - \left[2F'\left(De^{-h^2}\right) - \frac{2F\left(De^{-h^2}\right)}{De^{-h^2}} + \frac{1}{h^2}\frac{F\left(De^{-h^2}\right)}{De^{-h^2}} \right] w.
\end{aligned}$$

We rewrite this inequality as

$$\begin{aligned}
\Delta_f w - w_t \geq & -2kw + 2\left(2h - \frac{1}{h}\right)\langle \nabla w, \nabla h\rangle + 2\left(2 + \frac{1}{h^2}\right)w^2 + \frac{1}{h}\langle \nabla a, \nabla h\rangle \\
& - \frac{a}{h^2}w - \left[2F'(u) - \frac{2F(u)}{u} + \frac{1}{h^2}\frac{F(u)}{u} \right] w.
\end{aligned} \tag{4.12}$$

Since $h \geq 1$, from the Cauchy–Schwarz inequality, we get

$$\frac{1}{h}\langle \nabla a, \nabla h\rangle \leq \frac{1}{h}|\langle \nabla a, \nabla h\rangle| \leq |\nabla a||\nabla h| = |\nabla a| w^{\frac{1}{2}}. \tag{4.13}$$

In addition, we remark that

$$2kw + \frac{a}{h^2}w \leq 2\max\{k, 0\}w + \frac{1}{h^2}\max\{a, 0\}w \leq (2k^+ + a^+)\,w,$$

and

$$\begin{aligned}
& 2F'(u) - \frac{2F(u)}{u} + \frac{1}{h^2}\frac{F(u)}{u} \\
&= 2F'(u) - \frac{2F(u)}{u} + \frac{1}{1 + \ln B - \ln u}\frac{F(u)}{u} \\
&\leq \max\left\{ 2F'(u) - \frac{2F(u)}{u} + \frac{1}{1 + \ln B - \ln u}\frac{F(u)}{u}, 0 \right\} \\
&= \left[2F'(u) - \frac{2F(u)}{u} + \frac{1}{1 + \ln B - \ln u}\frac{F(u)}{u} \right]^+ = G.
\end{aligned} \tag{4.14}$$

These inequalities, combined with (4.12), yield that

$$\Delta_f w - w_t \geq 2\left(2h - \frac{1}{h}\right)\langle \nabla w, \nabla h\rangle + |\nabla a| w^{\frac{1}{2}} - (2k^+ + a^+ + G)\, w + 2\left(2 + \frac{1}{h^2}\right) w^2.$$

The proof is complete. □

To prove the main theorems in this section, we recall the following useful space-time cut-off function in [8, 36, ?, 97].

Lemma 4.2. *Given $\tau \in [0, T]$, there exists a smooth cut-off function $\overline{\psi}(r,t)$ supported in $[0, R] \times [0, T]$ satisfying following propositions*

(i) $0 \leq \overline{\psi}(r,t) \leq 1$ *in* $[0, R] \times [0, T]$.

(ii) *The equalities* $\overline{\psi}(r,t) = 1$ *and* $\frac{\partial \overline{\psi}}{\partial r}(r,t) = 0$ *hold in* $[0, \frac{R}{2}] \times [\tau, T]$ *and* $[0, \frac{R}{2}] \times [0, T]$, *respectively.*

(iii) *When* $0 < \epsilon < 1$, *there is a constant* C_ϵ *such that*

$$-\frac{C_\epsilon \overline{\psi}^\epsilon}{R} \leq \frac{\partial \overline{\psi}}{\partial r} \leq 0; \ \ and \ \left|\frac{\partial^2 \overline{\psi}}{\partial r^2}\right| \leq \frac{C_\epsilon \overline{\psi}^\epsilon}{R^2}.$$

(iv) $\overline{\psi}(r, 0) = 0$ *for all* $r \in [0, \infty)$ *and* $\left|\frac{\partial \overline{\psi}}{\partial t}\right| \leq \frac{c\overline{\psi}^{\frac{1}{2}}}{\tau}$ *on* $[0, \infty) \times [0, T]$ *for some* $c > 0$.

Now, we are ready to prove Theorem 4.1.

Proof of Theorem 4.1. With each fixed time $\tau \in (0, T]$, we choose cut-off function $\overline{\psi}(r,t)$ satisfying the conditions of Lemma 4.2. Our main goal is to prove that inequality (4.2) in Theorem 4.1 holds at every point (x, τ) in $Q_{\frac{R}{2},T}$. Since τ is arbitrary, the conclusion of Theorem 4.1 will immediately follow. To this purpose, we introduce a smooth cut-off function $\psi : M \times [0, T] \to \mathbb{R}$ by

$$\psi(x,t) := \overline{\psi}\left(r(x,t), t\right), \tag{4.15}$$

where $x_0 \in M$ is a fixed point given in the statement of Theorem 4.1 and $r(x,t) = \text{dist}(x, x_0, t)$ is the distance function from the fixed point $x_0 \in M$ at time t. Let (x_1, t_1) be a maximum point for the function ψw in the set $Q_{R,T}$. If $(\psi w)(x_1, t_1) \leq 0$ then $(\psi w)(x, \tau) \leq 0$ for all $x \in M$ such that $\text{dist}\,(x, x_0, \tau) \leq R$. Note that $\psi(x, \tau) \equiv 1$ for all $x \in M$ satisfying $\text{dist}\,(x, x_0, \tau) \leq \frac{R}{2}$. This implies that $w(x, \tau) \leq 0$ when $d\,(x, x_0, \tau) \leq \frac{R}{2}$. Since τ is arbitrarily chosen, we see that the

(4.2) holds on $\mathcal{Q}_{\frac{R}{2},T}$ in this case. Next, we consider the case $(\psi w)(x_1,t_1) > 0$. By the standard argument of Calabi [17], we may assume that (ψw) is smooth at (x_1,t_1). Obviously at (x_1,t_1), we have

$$\Delta_f(\psi w) \le 0, \quad \nabla(\psi w) = 0 \quad \text{and} \quad (\psi w)_t \ge 0.$$

Hence, still being at (x_1,t_1), we see that

$$0 \ge \Delta_f(\psi w) - (\psi w)_t = \psi(\Delta_f w - w_t) + w(\Delta_f \psi - \psi_t) + 2\langle \nabla w, \nabla \psi\rangle.$$

Using the fact that $0 = \nabla(\psi w) = w\nabla\psi + \psi\nabla w$, we get

$$0 \ge \psi(\Delta_f w - w_t) + w\Delta_f\psi - w\psi_t - 2\frac{|\nabla\psi|^2}{\psi}w$$

at (x_1,t_1). The above inequality, combined with (4.6), yield that

$$\begin{aligned}
0 \ge &-(2k^+ + a^+ + G)\psi w - |\nabla a|\psi w^{\frac{1}{2}} - 2\left(2h - \frac{1}{h}\right)\langle\nabla\psi, \nabla h\rangle w \\
&+ 2\left(2 + \frac{1}{h^2}\right)\psi w^2 + w\Delta_f\psi - w\psi_t - 2\frac{|\nabla\psi|^2}{\psi}w.
\end{aligned}$$

or equivalent to

$$\begin{aligned}
4\psi w^2 \le &\frac{2h^2}{1+2h^2}(2k^+ + a^+ + G)\psi w + \frac{4h(1-2h^2)}{1+2h^2}\langle\nabla h, \nabla\psi\rangle w \\
&+ \frac{2h^2}{1+2h^2}|\nabla a|\psi w^{\frac{1}{2}} - \frac{2h^2}{1+2h^2}w\Delta_f\psi + \frac{2h^2}{1+2h^2}w|\psi_t| + \frac{4h^2}{1+2h^2}\frac{|\nabla\psi|^2}{\psi}w
\end{aligned}$$

at (x_1,t_1). Note that

$$0 < \frac{2h^2}{1+2h^2} \le 1, \; 0 < \frac{2}{1+2h^2} \le 1 \text{ and } 0 < \frac{2h}{1+2h^2} \le 1.$$

Thus, from the above inequality, we have

$$\begin{aligned}
4\psi w^2 \le &(2k^+ + a^+ + G)\psi w + |\nabla a|\psi w^{\frac{1}{2}} + \frac{4h(1-2h^2)}{1+2h^2}\langle\nabla h, \nabla\psi\rangle w \\
&- \frac{2h^2}{1+2h^2}w\Delta_f\psi + \frac{2|\nabla\psi|^2}{\psi}w + |\psi_t|w
\end{aligned} \tag{4.16}$$

at (x_1,t_1). Now, we consider two possible cases.

Case 1. We assume $d(x_1,x_0,t_1) \ge 1$. Since $\mathrm{Ric}_f \ge -(n-1)K$ and $r(x_1,t_1) \ge 1$

in $\mathcal{Q}_{R,T}, R \geq 2$, we have the f-Laplace comparison theorem (see Theorem 3.1 in [103]) that

$$\Delta_f r(x_1, t_1) \leq \mu + (n-1)K(R-1) \tag{4.17}$$

where

$$\mu = \max_{(x,t)} \{\Delta_f r(x,t) : \text{dist}(x, x_0, t) = 1, 0 \leq t \leq T\}.$$

Next, we will estimate upper bounds for each term of the right-hand side (RHS) of (4.16). For simplicity, we let c denote a constant depending only on n, whose value may change from line to line.

For the first term on the RHS of (4.16), we have

$$\begin{aligned}
&(2k^+ + a^+ + G)\,\psi w + |\nabla a| w^{\frac{1}{2}}\psi \\
&= \left(\psi^{\frac{1}{2}} w\right)\left[\psi^{\frac{1}{2}}(2k^+ + a^+ + G)\right] + \left(\psi^{\frac{1}{4}} w^{\frac{1}{2}}\right)\left(\psi^{\frac{3}{4}}|\nabla a|\right) \\
&\leq \frac{1}{2}\psi w^2 + c\,(2k^+ + a^+ + G)^2 + \frac{1}{2}\psi w^2 + c\left(\psi^{\frac{3}{4}}|\nabla a|\right)^{\frac{4}{3}} \\
&\leq \psi w^2 + c\,(k^+)^2 + c\left(\sup_{\mathcal{Q}_{R,T}} G\right)^2 + c\left[(a^+)^2 + |\nabla a|^{\frac{4}{3}}\right].
\end{aligned}$$

Using the inequality $x^4 + y^4 \leq (x+y)^4$ for all $x, y \geq 0$, we see that

$$(a^+)^2 + |\nabla a|^{\frac{4}{3}} \leq \left[(a^+)^{\frac{1}{2}} + |\nabla a|^{\frac{1}{3}}\right]^4 \leq \left[\sup_{\mathcal{Q}_{R,T}}\left\{(a^+)^{\frac{1}{2}} + |\nabla a|^{\frac{1}{3}}\right\}\right]^4 = \Gamma_a^4.$$

This implies that

$$(2k^+ + a^+ + G)\,\psi w + |\nabla a| w^{\frac{1}{2}}\psi \leq \psi w^2 + c\,(k^+)^2 + c\left(\sup_{\mathcal{Q}_{R,T}} G\right)^2 + c\Gamma_a^4. \tag{4.18}$$

For the second term on the RHS of (4.16), we have

$$\begin{aligned}
\frac{4h(1-2h^2)}{1+2h^2}\langle \nabla h, \nabla\psi\rangle w &\leq 4h\frac{|1-2h^2|}{1+2h^2}|\nabla\psi|\,|\nabla h|\,w \\
&\leq 4h\,|\nabla\psi|\,w^{\frac{3}{2}} = 4(\psi w^2)^{\frac{3}{4}}\left(h\,|\nabla\psi|\,\psi^{-\frac{3}{4}}\right) \\
&\leq \frac{1}{2}\psi w^2 + ch^4\frac{|\nabla\psi|^4}{\psi^3} \leq \frac{1}{2}\psi w^2 + c\frac{A^2}{R^4},
\end{aligned} \tag{4.19}$$

where $A = 1 + \ln B - \ln\left(\inf_{\mathcal{Q}_{R,T}} u\right)$.

For the third term on the RHS of (4.16), using Lemma 4.2 and (4.17), we have

$$
\begin{aligned}
-\frac{2h^2}{1+2h^2} w\Delta_f\psi &\leq w\left[|\psi_r|\left(\mu^+ + (n-1)K(R-1)\right) + |\psi_{rr}|\right] \\
&\leq \left(\psi^{\frac{1}{2}} w\right)\left(\frac{|\psi_{rr}|}{\psi^{\frac{1}{2}}}\right) + \left(\mu^+ + (n-1)K(R-1)\right)\left(\psi^{\frac{1}{2}} w\right)\left(\frac{|\psi_r|}{\psi^{\frac{1}{2}}}\right) \\
&\leq \frac{1}{2}\psi w^2 + c\frac{|\psi_r|^2}{\psi} + c\frac{(\mu^+)^2|\psi_r|^2}{\psi} + c\frac{K^2(R-1)^2|\psi_r|^2}{\psi} \\
&\leq \frac{1}{2}\psi w^2 + \frac{c}{R^4} + \frac{c(\mu^+)^2}{R^2} + cK^2. \qquad (4.20)
\end{aligned}
$$

For the fourth term on the RHS of (4.16), we have

$$
\frac{2|\nabla\psi|^2}{\psi} w = 2\left(|\nabla\psi|^2\psi^{-\frac{3}{2}}\right)\left(\psi^{\frac{1}{2}} w\right) \leq \frac{1}{2}\psi w^2 + c\frac{|\nabla\psi|^4}{\psi^3} \leq \frac{1}{2}\psi w^2 + \frac{c}{R^4}. \qquad (4.21)
$$

Next, we will estimate an upper bound for the operator $\partial_t\psi$ in $\mathcal{Q}_{R,T}$. Fixed $t > 0$ and let $\xi = \xi(s) : [0, a] \to M$ be a minimal geodesic with respect to $g(x,t)$ connecting $x_0 = \xi(0)$ to $x = \xi(a)$. Then, we have

$$
\begin{aligned}
\partial_t r(x,t) = \partial_t d(x, x_0, t) &= \partial_t \int_0^a |\xi'(s)|_{g(t)}\, ds \\
&= \int_0^a \frac{(\partial_t g)(\xi', \xi')}{2|\xi'|_{g(t)}} ds \\
&\geq \int_0^a -H|\xi'|_{g(t)}\, ds \geq -Hr(x,t) \geq -HR.
\end{aligned}
$$

Combining this with propositions (iii) and (iv) of Lemma 4.2 we deduce that

$$
\partial_t\psi = \overline{\psi}_t + \overline{\psi}_r\partial_t r \leq \overline{\psi}_t - HR\overline{\psi}_r \leq \left[\frac{\left|\overline{\psi}_t\right|}{\overline{\psi}^{\frac{1}{2}}} + R\frac{H\left|\overline{\psi}_r\right|}{\overline{\psi}^{\frac{1}{2}}}\right]\overline{\psi}^{\frac{1}{2}} \leq c(1+\tau H)\frac{\overline{\psi}^{\frac{1}{2}}}{\tau}. \qquad (4.22)
$$

From this and the Cauchy-Schwarz inequality, we get

$$
\begin{aligned}
w|\psi_t| = \left(w\sqrt{\psi}\right)\frac{|\psi_t|}{\sqrt{\psi}} &\leq c(w\sqrt{\psi})(1+\tau H)\frac{\sqrt{\psi}}{\tau} \\
&\leq \frac{1}{2}\psi w^2 + c\frac{(1+\tau H)^2}{\tau^2} \leq \frac{1}{2}\psi w^2 + c\left(\frac{1}{\tau^2} + H^2\right). \qquad (4.23)
\end{aligned}
$$

Substituting (4.18)-(4.23) into (4.16), we conclude that

$$\psi w^2 \leq c\frac{A^2+1}{R^4} + \frac{c\left(\mu^+\right)^2}{R^2} + \frac{c}{\tau^2} + c\left(k^+\right)^2 + cK^2 + cH^2 + c\left(\sup_{\mathcal{Q}_{R,T}} G\right)^2 + c\Gamma_a^4 \tag{4.24}$$

at (x_1, t_1). Note that $0 \leq \psi \leq 1$ and $\psi(x,\tau) = 1$ when $d(x, x_0, \tau) \leq \frac{R}{2}$. Thus, we obtain

$$\begin{aligned} w^2(x,\tau) &\leq \left(\psi w^2\right)(x_1,t_1) \\ &\leq c\frac{A^2+1}{R^4} + \frac{c\left(\mu^+\right)^2}{R^2} + \frac{c}{\tau^2} + c\left(k^+\right)^2 + cK^2 + cH^2 + c\mathcal{P}^2 + c\Gamma_a^4 \end{aligned}$$

for all $x \in M$ such that $d(x, x_0, \tau) \leq \frac{R}{2}$, where $\mathcal{P} = \left(\sup_{\mathcal{Q}_{R,T}} G\right)$. Since $\tau \in (0, T]$ is arbitrary and $w = |\nabla h|^2$, this completes the proof of Theorem 4.1 in this case.

Case 2. We assume $d(x_1, x_0, t_1) \leq 1$. Then, by the definition of ψ, we see that ψ is a constant in space direction in $\mathcal{Q}_{\frac{R}{2},T}$ where $R \geq 2$. Thus, from (4.16), we have

$$4\psi w^2 \leq \left(2k^+ + a^+ + G\right)\psi w + |\nabla a|\psi w^{\frac{1}{2}} + w\left|\psi_t\right|. \tag{4.25}$$

Substituting (4.18) and (4.23) into (4.25), we obtain

$$\psi w^2 \leq \frac{c}{\tau^2} + c\left(k^+\right)^2 + cH^2 + c\mathcal{P}^2 + c\Gamma_a^4$$

at (x_1, t_1). Note that $0 \leq \psi \leq 1$ and $\psi(x,\tau) = 1$ when $d(x, x_0, \tau) \leq \frac{R}{2}$. Thus, we deduce that

$$\begin{aligned} w^2(x,\tau) &\leq \left(\psi w^2\right)(x_1,t_1) \\ &\leq \frac{c}{\tau^2} + c\left(k^+\right)^2 + cH^2 + c\mathcal{P}^2 + c\Gamma_a^4 \end{aligned}$$

for all $x \in M$ such that $d(x, x_0, \tau) \leq \frac{R}{2}$. From the definition of $h(x,t)$ and the above inequality, we complete the proof of Theorem 4.1 in this case. □

Remark 4.1. When $\left(M, g(x,t), e^{-f(x,t)}d\mu\right)_{t\in[0,T]}$ is a complete solution to the (k,m)-super Perelman-Ricci flow $(m < \infty)$, we can give another gradient estimate for the positive solutions to the general heat equation (4.1) in $\mathcal{Q}_{\frac{R}{2},T}$. Indeed, if $\mathrm{Ric}_f^m \geq -(m+n-1)K$ for some constant $K \geq 0$ then the generalized Laplacian comparison theorem [103] implies

$$\begin{aligned}\Delta_f r(x,t) &\le (m+n-1)\sqrt{K}\coth\left(\sqrt{K}r(x,t)\right)\\ &\le \frac{1}{r(x,t)}(m+n-1)\left[1+\sqrt{K}r(x,t)\right]\\ &= (m+n-1)\left(\frac{1}{r(x,t)}+\sqrt{K}\right).\end{aligned}\tag{4.26}$$

From this and the definition of the function $\overline{\psi}$, we see that

$$\begin{aligned}\Delta_f\psi &= \overline{\psi}_r\Delta_f r+\overline{\psi}_{rr}|\nabla r|^2\\ &\ge -\frac{C_{\frac12}\psi^{\frac12}}{R}(m+n-1)\left(\frac{2}{R}+\sqrt{K}\right)-\frac{C_{\frac12}\psi^{\frac12}}{R^2}\\ &\ge -\frac{C_{\frac12}\psi^{\frac12}(m+n)\left(2+R\sqrt{K}\right)}{R^2}.\end{aligned}\tag{4.27}$$

for some positive constant $C_{\frac12}$ in $\mathcal{Q}_{R,T}$, $R\ge 2$. Using the above inequality and repeating arguments in the proof of Theorem 4.1, we get the following result.

Theorem 4.4. *Let* $\left(M,g(x,t),e^{-f(x,t)}d\mu\right)_{t\in[0,T]}$ *be a complete solution to the* (k,m)*-super Perelman-Ricci flow* (1.10) *and* u *be a smooth positive solution to the non-linear heat equation* (4.1) *in* $\mathcal{Q}_{R,T}$. *Assume that* $0<u\le B$ *and*

$$\mathrm{Ric}_f^m \ge -(m+n-1)K, \quad \frac{\partial g}{\partial t}\ge -2Hg$$

for some $K,H\ge 0$ *in* $\mathcal{Q}_{R,T}$. *Then there exists a constant* c *depending only* n *such that*

$$\frac{|\nabla u|}{u}\le c\left[\frac{\sqrt{A}}{R}+\frac{1}{\sqrt{t}}+\sqrt[4]{\left(k^+\right)^2+K^2+H^2+\mathcal{P}^2}+\Gamma_a\right]\sqrt{1+\ln\frac{B}{u}}\tag{4.28}$$

for all $(x,t)\in\mathcal{Q}_{\frac{R}{2},T}$ *with* $t\ne 0$, *where* $A,\Gamma_a,\mathcal{P}$ *are the same as Theorem 4.1.*

Remark 4.2. When $k=m=0$, the (k,m)-super Perelman-Ricci flow returns the super Ricci flow. Furthermore, if we have $|\mathrm{Ric}(x,t)|\le\kappa$ for some constant $\kappa\ge 0$ then

$$\frac{\partial}{\partial t}g(x,t)\ge -2\,\mathrm{Ric}(x,t)\ge -2\kappa g(x,t)$$

in $\mathcal{Q}_{R,T}$. From this point of view and Theorem 4.4, we obtain the following local gradient estimate for the nonlinear heat equation under super Ricci flow.

Theorem 4.5. *Let $(M, g(x,t))_{t\in[0,T]}$ be a complete solution to the super Ricci flow (1.4) and u be a smooth positive solution to the nonlinear heat equation*

$$\left(\frac{\partial}{\partial t} - a(x,t) - \Delta\right) u(x,t) = F(u(x,t)) \tag{4.29}$$

in the set $\mathcal{Q}_{R,T}$, where $a(x,t)$ is a function which is $\mathcal{C}^2$ in the x-variable and $\mathcal{C}^1$ in the t-variable, and $F(u)$ is a $\mathcal{C}^2$ function of u. Assume that $|\mathrm{Ric}(x,t)| \leq \kappa$ for some constant $\kappa \geq 0$ for all $(x,t) \in \mathcal{Q}_{R,T}$. If $u(x,t) \leq B$ for some constant $B > 0$ in $\mathcal{Q}_{R,T}$, then there exists a constant c depending only n such that

$$\frac{|\nabla u|}{u} \leq c\left[\frac{\sqrt{A}}{R} + \frac{1}{\sqrt{t}} + \sqrt{\kappa} + \sqrt{\mathcal{P}} + \Gamma_a\right]\sqrt{1 + \ln\frac{B}{u}} \tag{4.30}$$

for all $(x,t) \in \mathcal{Q}_{\frac{R}{2},T}$ with $t \neq 0$, where $A, \Gamma_a, \mathcal{P}$ are the same as Theorem 4.1.

As an application of Theorem 4.1, we can drive a global gradient estimate for positive bounded solutions to (4.1).

Corollary 4.1. *Let $\left(M, g(x,t), e^{-f(x,t)}d\mu\right)_{t\in[0,T]}$ be a complete solution to the (k,∞)-super Perelman-Ricci flow (1.11) and $u : M \times [0,T] \to \mathbb{R}$ be a smooth positive solution to the nonlinear heat equation (4.1). Assume that*

$$\mathrm{Ric}_f \geq -(n-1)K, \quad \frac{\partial g}{\partial t} \geq -2Hg$$

for some $K, H \geq 0$ on $M \times [0,T]$. If $\delta \leq u(x,t) \leq B$ for some constants $\delta, B > 0$, then there exists a constant c depending only n such that

$$\frac{|\nabla u|}{u} \leq c\left[\frac{1}{\sqrt{t}} + \sqrt[4]{(k^+)^2 + K^2 + H^2 + \mathcal{P}^2} + \Gamma_a\right]\sqrt{1 + \ln\frac{B}{u}}$$

on $M \times [0,T]$ with $t \neq 0$, where

$$\Gamma_a = \sup_{M\times[0,T]}\left\{(a^+)^{\frac{1}{2}} + |\nabla a|^{\frac{1}{3}}\right\},$$

and

$$\mathcal{P} = \sup_{M\times[0,T]}\left\{\left[2F'(u) - \frac{2F(u)}{u} + \frac{1}{1 + \ln B - \ln u}\frac{F(u)}{u}\right]^+\right\}.$$

Proof of Corollary 4.1. Since $\delta \leq u(x,t) \leq B$ is a smooth solution of the equation

(4.1), using Theorem 4.1, we get

$$\frac{|\nabla u|}{u} \leq c\left[\frac{\sqrt{A}}{R}+\sqrt{\frac{\mu^{+}}{R}}+\frac{1}{\sqrt{t}}+\sqrt[4]{\left(k^{+}\right)^{2}+K^{2}+H^{2}+\mathcal{P}^{2}}+\Gamma_{a}\right] \sqrt{1+\ln \frac{B}{u}}, \tag{4.31}$$

where $A=1+\ln B-\ln u \leq 1+\ln B-\ln \delta<\infty$. We notice that the inequality (4.31) holds for every $R \geq 2$, and each term of the right-hand side of (4.31) does not depend on R. Thus, the conclusion immediately follows by taking $R \rightarrow \infty$ in (4.31). The proof is complete. □

Another application of Theorem 4.1 is a Harnack-type inequality for positive solutions of (4.1), which can be used to compare solutions at the same time.

Corollary 4.2. *Under the same assumption as in Corollary 4.1, if $u: M \times [0,T] \rightarrow \mathbb{R}$ is a smooth solution to the general heat equation (4.1) and $\delta \leq u\left(x,t\right) \leq B$ for some positive constants δ, B, then for any $x_1, x_2 \in M$ and $t \in (0,T]$ we have*

$$\frac{u\left(x_{2}, t\right)}{B e} \leq\left[\frac{u\left(x_{1}, t\right)}{B e}\right]^{\beta^{2}} \tag{4.32}$$

where

$$\beta=\exp \left\{-c \operatorname{dist}\left(x_{1}, x_{2}, t\right)\left[\frac{1}{\sqrt{t}}+\sqrt[4]{\left(k^{+}\right)^{2}+K^{2}+H^{2}+\mathcal{P}^{2}}+\Gamma_{a}\right]\right\},$$

and $\Gamma_a, \mathcal{P}$ are the same as Corollary 4.1. Here c is a constant depending only on n.

Proof of Corollary 4.2. Let $\gamma(s)$ be a geodesic of minimal length with respect to the metric $g=g(x,t)$ connecting x_1 and $x_2, \gamma:[0,1] \rightarrow M, \gamma(0)=x_2, \gamma(1)=x_1$. We can assume that $\Gamma_a, \mathcal{P}<\infty$, otherwise $\beta=0$ and the inequality (4.32) is trivially true. Next, put $h=\sqrt{1+\ln \frac{B}{u}}=\sqrt{\ln \frac{D}{u}}$, where $D=Be$. Using the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}
\ln \frac{h\left(x_{1}, t\right)}{h\left(x_{2}, t\right)} &=\int_{0}^{1} \frac{d \ln (h(\gamma(s), t))}{d s} d s \\
&=\int_{0}^{1} \frac{\left\langle\nabla h(\gamma(s), t), \gamma^{\prime}(s)\right\rangle}{h(\gamma(s), t)} d s \\
& \leq \int_{0}^{1} \frac{|\nabla h|}{h}\left|\gamma^{\prime}\right| d s=\int_{0}^{1} \frac{|\nabla u|}{2 u\left(1+\ln \frac{B}{u}\right)}\left|\gamma^{\prime}\right| d s .
\end{aligned} \tag{4.33}$$

Here, we have used the fact that

$$\frac{|\nabla h|}{h}=\frac{\left|\nabla\sqrt{\ln\frac{D}{u}}\right|}{\sqrt{\ln\frac{D}{u}}}=\frac{|\nabla u|}{2u\ln\frac{D}{u}}=\frac{|\nabla u|}{2u\left(1+\ln\frac{B}{u}\right)}.$$

By the inequality $\sqrt{1+\ln\frac{B}{u}}\leq 1+\ln\frac{B}{u}$ and Corollary 4.1, we see that

$$\frac{|\nabla u|}{2u\left(1+\ln\frac{B}{u}\right)}\leq c\left(\frac{1}{\sqrt{t}}+\sqrt[4]{\left(k^{+}\right)^{2}+K^{2}+H^{2}+\mathcal{P}^{2}}+\Gamma_{a}\right),$$

where c is a constant depending only on n. This and (4.33) entail that

$$\ln\frac{h\left(x_1,t\right)}{h\left(x_2,t\right)}\leq c\,\mathrm{dist}\left(x_1,x_2,t\right)\left[\frac{1}{\sqrt{t}}+\sqrt[4]{\left(k^{+}\right)^{2}+K^{2}+H^{2}+\mathcal{P}^{2}}+\Gamma_{a}\right]$$

We rewrite this inequality as

$$\frac{h\left(x_1,t\right)}{h\left(x_2,t\right)}\leq\exp\left\{c\mathrm{dist}\left(x_1,x_2,t\right)\left[\frac{1}{\sqrt{t}}+\sqrt[4]{\left(k^{+}\right)^{2}+K^{2}+H^{2}+\mathcal{P}^{2}}+\Gamma_{a}\right]\right\}=\frac{1}{\beta}.$$

From the above inequality, by some easy calculations, we obtain (4.32). □

4.2 Gradient estimates for (4.1) under the Yamabe flow

Let $\left(M,g(x,t),e^{-f(x,t)}d\mu\right)_{t\in[0,T]}$ be a complete solution to the Yamabe flow (1.13). Our goal in this section is to drive a local Hamilton type gradient estimate for any positive bounded solutions of the nonlinear heat equation (4.1) under the Yamabe flow (1.13). Assume that $0<u(x,t)\leq B$ for some positive constant B, is a smooth solution to the equation (4.1) in $\mathcal{Q}_{R,T}$. Then we see that u and g solve

$$\begin{cases} u_t=\Delta_f u+au+F(u),\\ \dfrac{\partial g}{\partial t}=-Sg. \end{cases}\tag{4.34}$$

Consider the function $h=\sqrt{1+\ln\frac{B}{u}}=\sqrt{\ln\frac{D}{u}}\geq 1$ in $\mathcal{Q}_{R,T}$, where $D=Be$. From (4.5), we see that

$$\left(\Delta_f-\partial_t\right)h=-|\nabla h|^2\left(\frac{1}{h}-2h\right)+\frac{a}{2h}+\frac{F\left(De^{-h^2}\right)}{2Dhe^{-h^2}}.\tag{4.35}$$

To prove Theorem 4.3, we need the following useful lemma.

Lemma 4.3. *Under the same assumption as in Theorem 4.3, for all (x,t) in $\mathcal{Q}_{R,T}$, the function $w = |\nabla h|^2$ satisfies*

$$\Delta_f w - w_t \geq 2\left(2h - \frac{1}{h}\right)\langle \nabla w, \nabla h\rangle - |\nabla a| w^{\frac{1}{2}} \\ - [(n-1)K + \mathcal{H} + G + a^+]w + 2\left(2 + \frac{1}{h^2}\right)w^2, \tag{4.36}$$

where G is the same as Lemma 4.1.

Proof of Lemma 4.3. By the Bochner-Weitzenböck formula and note that $\mathrm{Ric}_f \geq -(n-1)K$, we get

$$\frac{1}{2}\Delta_f w = \frac{1}{2}\Delta_f |\nabla h|^2 = \left|\nabla^2 h\right|^2 + \mathrm{Ric}_f(\nabla h, \nabla h) + \langle \nabla \Delta_f h, \nabla h\rangle \\ \geq -(n-1)K|\nabla h|^2 + \langle \nabla \Delta_f h, \nabla h\rangle . \tag{4.37}$$

From (3.6) and (4.34), we obtain

$$w_t = -\frac{\partial g}{\partial t}(\nabla h, \nabla h) + 2\langle \nabla (h_t), \nabla h\rangle = S|\nabla h|^2 + 2\langle \nabla (h_t), \nabla h\rangle .$$

Combining this with (4.37), we imply that

$$\Delta_f w - w_t \geq -2(n-1)K|\nabla h|^2 + 2\langle \nabla \Delta_f h, \nabla h\rangle - S|\nabla h|^2 - 2\langle \nabla (h_t), \nabla h\rangle \\ = -[2(n-1)K + S]w + 2\langle \nabla (\Delta_f h - h_t), \nabla h\rangle .$$

This and (4.35) entail that

$$\Delta_f w - w_t \\ \geq -[2(n-1)K + S]w + 2\left(2h - \frac{1}{h}\right)\langle \nabla (|\nabla h|^2), \nabla h\rangle \\ + 2|\nabla h|^2\left\langle \nabla\left(2h - \frac{1}{h}\right), \nabla h\right\rangle + \left\langle \nabla\left(\frac{a}{h}\right), \nabla h\right\rangle + \left\langle \nabla\left(\frac{F\left(De^{-h^2}\right)}{Dhe^{-h^2}}\right), \nabla h\right\rangle .$$

Substituting (4.9)-(4.11) into the above inequality, we conclude that

$$\Delta_f w - w_t \geq -[2(n-1)K + S]w + 2\left(2h - \frac{1}{h}\right)\langle \nabla w, \nabla h\rangle + 2\left(2 + \frac{1}{h^2}\right)w^2 \\ + \frac{1}{h}\langle \nabla a, \nabla h\rangle - \frac{a}{h^2}w - \left[2F'(u) - \frac{2F(u)}{u} + \frac{1}{h^2}\frac{F(u)}{u}\right]w. \tag{4.38}$$

As in the proof of Lemma 4.1, we have

$$\frac{a}{h^2}w \leq \frac{1}{h^2}\max\{a,0\}w \leq a^+w, \quad \frac{1}{h}\langle\nabla a, \nabla h\rangle \leq |\nabla a||\nabla h| = |\nabla a|w^{\frac{1}{2}},$$

and

$$\begin{aligned}
&2F'(u) - \frac{2F(u)}{u} + \frac{1}{h^2}\frac{F(u)}{u} \\
&\leq \max\left\{2F'(u) - \frac{2F(u)}{u} + \frac{1}{1+\ln B - \ln u}\frac{F(u)}{u}, 0\right\} \\
&= \left[2F'(u) - \frac{2F(u)}{u} + \frac{1}{1+\ln B - \ln u}\frac{F(u)}{u}\right]^+ = G.
\end{aligned}$$

Plugging these above inequalities into (4.38) and note that $S \leq \mathcal{H}$, we complete the proof of Lemma 4.3. □

In the following, we will use the same arguments as in the proof of Theorem 4.1 to prove Theorem 4.3. Specifically, we will apply Lemma 4.3 and the localized technique to obtain an upper bound for the function w^2 in $\mathcal{Q}_{\frac{R}{2},T}$.

Proof of Theorem 4.3. For the fixed $\tau \in (0,T]$, let (x_1,t_1) be a maximum point for the function ψw in the set $\mathcal{Q}_{R,T}$, where the function ψ is defined as in (4.15). We may suppose that $(\psi w)(x_1,t_1) > 0$; otherwise if $(\psi w)(x_1,t_1) \leq 0$ then $(\psi w)(x,\tau) \leq 0$ for all $x \in M$ such that $\mathrm{dist}\,(x,x_0,t) \leq R$; here, we used fact that (x_1,t_1) is a maximum point of ψw in $\mathcal{Q}_{R,T}$. Note that $\psi(x,\tau) \equiv 1$ for all $x \in M$ satisfying $\mathrm{dist}\,(x,x_0,\tau) \leq \frac{R}{2}$. It shows that

$$w(x,\tau) \leq 0 \quad \text{when} \quad \mathrm{dist}\,(x,x_0,\tau) \leq \frac{R}{2}.$$

Since τ is arbitrarily chosen, we see that the inequality (4.2) holds on $\mathcal{Q}_{\frac{R}{2},T}$. Moreover, we have $t_1 \neq 0$, since $(\psi w)(x_1,t_1) > 0$. According to the standard argument of Calabi [17], we may assume that (ψw) is smooth at (x_1,t_1). Clearly, at (x_1,t_1), we have

$$\Delta_f(\psi w) \leq 0, \nabla(\psi w) = 0 \text{ and } (\psi w)_t \geq 0.$$

From these results, we see that

$$0 \geq \psi(\Delta_f w - w_t) + w\Delta_f\psi - w\psi_t - 2\frac{|\nabla\psi|^2}{\psi}w$$

at (x_1, t_1). The above inequality combined with (4.36) yield that

$$4\psi w^2 \leq \frac{2h^2}{1+2h^2}\left[(n-1)K + \mathcal{H} + G + a^+\right]\psi w - \frac{4h\left(1-2h^2\right)}{1+2h^2}\langle \nabla h, \nabla \psi\rangle w + \frac{2h}{1+2h^2}|\nabla a|\psi w^{\frac{1}{2}} - \frac{2h^2}{1+2h^2} w\Delta_f \psi + \frac{2h^2}{1+2h^2} w\left|\psi_t\right| + \frac{4h^2}{1+2h^2}\frac{|\nabla\psi|^2}{\psi} w.$$

at (x_1, t_1). Since $h \geq 1$, we have

$$0 < \frac{2h^2}{1+2h^2} \leq 1,\ 0 < \frac{2}{1+2h^2} \leq 1, \text{ and } 0 < \frac{2h}{1+2h^2} \leq 1.$$

Using these inequalities, we get

$$4\psi w^2 \leq \left[(n-1)K + \mathcal{H} + G + a^+\right]\psi w + |\nabla a|\psi w^{\frac{1}{2}} + \left|\psi_t\right| w - \frac{4h\left(1-2h^2\right)}{1+2h^2}\langle \nabla h, \nabla\psi\rangle w - \frac{2h^2}{1+2h^2} w\Delta_f\psi + \frac{2|\nabla\psi|^2}{\psi} w. \tag{4.39}$$

at (x_1, t_1). We now consider two possible cases.

Case 1. We assume $d(x_1, x_0, t_1) \geq 1$. Since $\mathrm{Ric}_f \geq -(n-1)K$ and $r(x_1, t_1) \geq 1$ in $\mathcal{Q}_{R,T}$, $R \geq 2$, we have the f-Laplace comparison theorem (see Theorem 3.1 in [103]) that $\Delta_f r(x_1, t_1) \leq \mu + (n-1)K(R-1)$ where

$$\mu = \max_{(x,t)}\left\{\Delta_f r(x,t) : \mathrm{dist}\,(x, x_0, t) = 1, 0 \leq t \leq T\right\}.$$

Next, we will estimate upper bounds for each term of the right-hand side of (4.39). For simplicity, we let c denote a constant depending only on n, whose value may change from line to line. Performing similar arguments as in the proof of Theorem 4.1, we have the following inequalities

$$\begin{cases} \dfrac{2|\nabla\psi|^2}{\psi} w \leq \dfrac{1}{2}\psi w^2 + \dfrac{c}{R^4}, \\ -\dfrac{4h\left(1-2h^2\right)}{1+2h^2}\langle\nabla h, \nabla\psi\rangle w \leq \dfrac{1}{2}\psi w^2 + c\dfrac{A^2}{R^4}, \\ -\dfrac{2h^2}{1+2h^2} w\Delta_f\psi \leq \dfrac{1}{2}\psi w^2 + \dfrac{c}{R^4} + \dfrac{c\left(\mu^+\right)^2}{R^2} + cK^2, \end{cases} \tag{4.40}$$

where $A = 1 + \ln B - \ln\left(\inf_{\mathcal{Q}_{R,T}} u\right)$. Moreover, we also have

$$\begin{aligned}
&[2(n-1)K + \mathcal{H} + G + a^+]\,\psi w + |\nabla a| w^{\frac{1}{2}}\psi \\
&= \left(\psi^{\frac{1}{2}} w\right)\psi^{\frac{1}{2}}\left[2(n-1)K + \mathcal{H} + G + a^+\right] + \left(\psi^{\frac{1}{4}} w^{\frac{1}{2}}\right)\left(\psi^{\frac{3}{4}}|\nabla a|\right) \\
&\leq \frac{1}{2}\psi w^2 + c\left[2(n-1)K + \mathcal{H} + G + a^+\right]^2 + \frac{1}{2}\psi w^2 + c\left(\psi^{\frac{3}{4}}|\nabla a|\right)^{\frac{4}{3}} \\
&\leq \psi w^2 + cK^2 + c\mathcal{H}^2 + cG^2 + c\left[\left(a^+\right)^2 + |\nabla a|^{\frac{4}{3}}\right] \\
&\leq \psi w^2 + cK^2 + c\mathcal{H}^2 + c\mathcal{P}^2 + c\Gamma_a^4, \qquad (4.41)
\end{aligned}$$

where $\mathcal{P} = \left(\sup_{\mathcal{Q}_{R,T}} G\right)$ and $\Gamma_a = \sup_{\mathcal{Q}_{R,T}}\left\{\left(a^+\right)^{\frac{1}{2}} + |\nabla a|^{\frac{1}{3}}\right\}$. Next, fixed $t > 0$ and let $\xi = \xi(s) : [0, a] \to M$ be a minimal geodesic with respect to $g(x,t)$ connecting $x_0 = \xi(0)$ to $x = \xi(a)$. Then, we have

$$\begin{aligned}
\partial_t r(x,t) = \partial_t d\left(x, x_0, t\right) &= \partial_t \int_0^a \left|\xi'(s)\right|_{g(t)} ds \\
&= -\frac{1}{2}\int_0^a S\left|\xi'(s)\right|_{g(t)} ds \\
&\geq -\frac{1}{2}\int_0^a \mathcal{H}\left|\xi'(s)\right|_{g(t)} ds \geq -\frac{1}{2}\mathcal{H} r(x,t) \geq -\frac{1}{2}\mathcal{H} R.
\end{aligned}$$

Combining this with propositions iii and iv of Lemma 4.2, we conclude that

$$\begin{aligned}
\partial_t \psi = \overline{\psi}_t + \overline{\psi}_r \partial_t r &\leq \overline{\psi}_t - \frac{1}{2}\mathcal{H} R \overline{\psi}_r \\
&\leq \left[\frac{\left|\overline{\psi}_t\right|}{\overline{\psi}^{\frac{1}{2}}} + \frac{\mathcal{H} R}{2}\frac{\left|\overline{\psi}_r\right|}{\overline{\psi}^{\frac{1}{2}}}\right]\overline{\psi}^{\frac{1}{2}} \leq c(1 + \tau\mathcal{H})\frac{\overline{\psi}^{\frac{1}{2}}}{\tau}.
\end{aligned}$$

Using the above inequality and the Cauchy-Schwarz inequality, we get

$$\begin{aligned}
w\left|\psi_t\right| = \left(w\sqrt{\psi}\right)\frac{\left|\psi_t\right|}{\sqrt{\psi}} &\leq c\left(w\sqrt{\psi}\right)(1 + \tau\mathcal{H})\frac{\sqrt{\psi}}{\tau} \\
&\leq \frac{1}{2}\psi w^2 + c\frac{(1 + \tau\mathcal{H})^2}{\tau^2} \leq \frac{1}{2}\psi w^2 + c\left(\frac{1}{\tau^2} + \mathcal{H}^2\right).
\end{aligned} \qquad (4.42)$$

Substituting (4.40), (4.41) and (4.42) into (4.39), we conclude that

$$\psi w^2 \leq c\frac{A^2 + 1}{R^4} + \frac{c\left(\mu^+\right)^2}{R^2} + \frac{c}{\tau^2} + cK^2 + c\mathcal{H}^2 + c\mathcal{P}^2 + c\Gamma_a^4$$

at (x_1,t_1). Since $0\le\psi\le1$ and $\psi(x,\tau)=1$ when $d(x,x_0,\tau)\le\frac{R}{2}$, we get

$$w^2(x,\tau)\le\left(\psi w^2\right)(x_1,t_1)\le c\frac{A^2+1}{R^4}+\frac{c\left(\mu^+\right)^2}{R^2}+\frac{c}{\tau^2}+cK^2+c\mathcal{H}^2+c\mathcal{P}^2+c\Gamma_a^4$$

for all $x\in M$ such that $d(x,x_0,\tau)\le\frac{R}{2}$. Since $\tau\in(0,T]$ is arbitrary and $w=|\nabla h|^2$, this completes the proof of Theorem 4.3 in this case.

Case 2. We assume $d(x_1,x_0,t_1)\le1$. Then, by the definition of ψ, we see that ψ is a constant in space direction in $\mathcal{Q}_{\frac{R}{2},T}$ where $R\ge2$. Thus, from (4.39), we have

$$4\psi w^2\le\left[(n-1)K+\mathcal{H}+G+a^+\right]\psi w+|\nabla a|\psi w^{\frac{1}{2}}+w\left|\psi_t\right|. \tag{4.43}$$

Substituting (4.41) and (4.42) into (4.43), we deduce that

$$\psi w^2\le\frac{c}{\tau^2}+cK^2+c\mathcal{H}^2+c\mathcal{P}^2+c\Gamma_a^4$$

at (x_1,t_1). Note that $0\le\psi\le1$ and $\psi(x,\tau)=1$ when $d(x,x_0,\tau)\le\frac{R}{2}$. Therefore,

$$w^2(x,\tau)\le\left(\psi w^2M\right)(x_1,t_1)\le\frac{c}{\tau^2}+cK^2+c\mathcal{H}^2+c\mathcal{P}^2+c\Gamma_a^4.$$

for all $x\in M$ such that $d(x,x_0,\tau)\le\frac{R}{2}$. From the definition of $h(x,t)$ and the above inequality, we complete the proof of Theorem 4.3 in this case. □

Applying the same technique as in Corollary 4.2, we get the following Harnack-type inequality.

Corollary 4.3. *Let $\left(M,g(x,t),e^{-f(x,t)}d\mu\right)_{t\in[0,T]}$ be a complete solution to the Yamabe flow (1.13) and $u:M\times[0,T]\to\mathbb{R}$ be a smooth positive solution to the semilinear heat equation (4.1). Assume that $\mathrm{Ric}_f\ge-(n-1)K$ and $R\le\mathcal{H}$ for some $K,\mathcal{H}\ge0$ on $M\times[0,T]$. If $\delta\le u(x,t)\le B$ for some $\delta,B>0$ then for any $x_1,x_2\in M$ and $t\in(0,T]$ we have*

$$\frac{u(x_2,t)}{Be}\le\left[\frac{u(x_1,t)}{Be}\right]^{\Sigma^2}$$

where

$$\Sigma=\exp\left\{-c\mathrm{dist}(x_1,x_2,t)\left[\frac{1}{\sqrt{t}}+\sqrt[4]{K^2+\mathcal{H}^2+\mathcal{P}^2}+\Gamma_a\right]\right\},$$

$\Gamma_a,\mathcal{P}$ are the same as Corollary 4.1, and c is a constant depending only on n.

4.3 Liouville type theorems and gradient estimates for some important geometric partial differential equations

4.3.1 On the nonlinear elliptic equations related to gradient Ricci solitons

Let $(M, g, e^{-f}d\mu)$ be an n-dimensional complete smooth metric measure space. In this subsection, we will study gradient estimates and related problems for positive smooth solutions of the following nonlinear elliptic equation

$$\Delta_f u(x) + a(x)u(x) + bu(x)\ln u(x) = 0 \tag{4.44}$$

and its parabolic counterpart

$$u_t(x,t) = \Delta_f u(x,t) + a(x,t)u(x,t) + bu(x,t)\ln u(x,t), \tag{4.45}$$

on $(M, g, e^{-f}d\mu)$. Here, $b \in \mathbb{R}$, $a(x)$ is a $\mathcal{C}^1$ function of x in (4.44), and $a(x,t)$ is a function which is $\mathcal{C}^2$ in the x-variable and $\mathcal{C}^1$ in the t-variable in (4.45).

Assume that the Bakry-Émery Ricci Ric_f is bounded below, we now apply Theorem 4.2 to derive a local gradient estimate for the equation (4.45) on the static smooth metric measure space $(M, g, e^{-f}d\mu)$. For $F(u) = bu\ln u$, we have

$$2F'(u) - \frac{2F(u)}{u} + \frac{1}{h^2}\frac{F(u)}{u} = 2(b\ln u + b) - 2b\ln u + \frac{1}{h^2}b\ln u = \frac{b}{h^2}\left(2h^2 + \ln u\right),$$

where $h = \sqrt{1 + \ln\frac{B}{u}} \geq 1$. Thus, we obtain

$$\mathcal{P} \leq \sup_{Q_{R,T}}\left\{[b(2 + 2\ln B - \ln u)]^+\right\} = \mathcal{P}_0.$$

From this and Theorem 4.2, we get the following gradient estimate result for positive solutions of the equation (4.45).

Theorem 4.6. *Let $(M, g, e^{-f}d\mu)$ be an n-dimensional complete smooth metric measure space with $\mathrm{Ric}_f \geq -(n-1)K$ for some constant $K \geq 0$ in $B(x_0, R)$. Assume that $0 < u(x,t) \leq B$ for some constant B, is a smooth solution to the nonlinear parabolic equation (4.45) in $Q_{R,T}$. Then there exists a constant c*

depending only n such that

$$\frac{|\nabla u|}{u} \leq c\left[\frac{\sqrt{A}}{R}+\sqrt{\frac{\mu^{+}}{R}}+\frac{1}{\sqrt{t}}+\sqrt{K}+\sqrt{\mathcal{P}_0}+\Gamma_a\right]\sqrt{1+\ln\frac{B}{u}} \tag{4.46}$$

in $Q_{\frac{R}{2},T}$ with $t \neq 0$, where $A = 1+\ln B - \ln\left(\inf_{Q_{R,T}} u\right)$ and

$$\mathcal{P}_0 = \sup\nolimits_{Q_{R,T}}\left\{[b(2+2\ln B-\ln u)]^{+}\right\}, \quad \Gamma_a = \sup\nolimits_{Q_{R,T}}\left\{(a^{+})^{\frac{1}{2}}+|\nabla a|^{\frac{1}{3}}\right\}.$$

When $b = 0$, the equation (4.45) becomes the weighted Schrödinger equation on $(M, g, e^{-f}d\mu)$. Using Theorem 4.6, we can derive the following result.

Corollary 4.4. *Under the same assumption as in Theorem 4.6, if $0 < u(x,t) \leq B$ for some constant $B > 0$ is a smooth solution to the weighted Schrödinger equation $u_t = \Delta_f u + au$ in $Q_{R,T}$, then there exists a constant c depending only n such that*

$$\frac{|\nabla u|}{u} \leq c\left[\frac{\sqrt{A}}{R}+\sqrt{\frac{\mu^{+}}{R}}+\frac{1}{\sqrt{t}}+\sqrt{K}+\sup_{Q_{R,T}}\left\{(a^{+})^{\frac{1}{2}}+|\nabla a|^{\frac{1}{3}}\right\}\right]\sqrt{1+\ln\frac{B}{u}}$$

in $Q_{\frac{R}{2},T}$ with $t \neq 0$, where A is the same as Theorem 4.6.

Remark 4.3. We notice that $\sqrt{1+\ln\frac{B}{u}} \leq 1+\ln\frac{B}{u}$ and

$$\sup_{Q_{R,T}}\left\{(a^{+})^{\frac{1}{2}}+|\nabla a|^{\frac{1}{3}}\right\} \leq \sup_{Q_{R,T}}\left\{(a^{+})^{\frac{1}{2}}\right\}+\sup_{Q_{R,T}}\left\{|\nabla a|^{\frac{1}{3}}\right\}.$$

From the above inequalities, we see that Corollary 4.4 is better than Theorem 1.1 in [116]. Moreover, our result can be regarded as an extension and improvement of Ruan [84].

On the other hand, if $a(x,t) \equiv 0$ then the equation (4.45) becomes the following general f-heat equation

$$u_t = \Delta_f u + bu\ln u. \tag{4.47}$$

From Theorem 4.6, we obtain the following gradient estimate result.

Corollary 4.5. *Under the same assumption as in Theorem 4.6, if $0 < u(x,t) \leq B$ for some constant $B > 0$ is a smooth solution to the equation (4.47) in $Q_{R,T}$, then there exists a constant c depending only n such that*

$$\frac{|\nabla u|}{u} \leq c\left[\frac{\sqrt{A}}{R}+\sqrt{\frac{\mu^{+}}{R}}+\frac{1}{\sqrt{t}}+\sqrt{K}+\sqrt{\sup_{Q_{R,T}}\left\{[b(2+2\ln B-\ln u)]^{+}\right\}}\right]\sqrt{1+\ln\frac{B}{u}}$$

in $Q_{\frac{R}{2},T}$ with $t \neq 0$, where A is the same as Theorem 4.6.

Remark 4.4. By the inequality $\ln(1+x) \leq x$ for all $x \geq 0$, we see that

$$\sqrt{1+\ln\frac{B}{u}} \leq \sqrt{\frac{B}{u}}.$$

Then, we can rewrite the inequality in Corollary 4.5 in the case $1 \leq u \leq B$ as

$$\frac{|\nabla u|}{\sqrt{u}} \leq cB\left[\frac{\sqrt{1+\ln B}}{R} + \sqrt{\frac{\mu^+}{R}} + \frac{1}{\sqrt{t}} + \sqrt{K} + \sqrt{\sup_{Q_{R,T}}\{[b(2+2\ln B - \ln u)]^+\}}\right].$$

This shows that Corollary 4.5 is better than Theorem 1.3 of Jiang [50] and Theorem 1.1 of Wu [105] in the case $a < 0$.

An immediate application of Theorem 4.6 is the following Liouville type result for posstive solutions of the equation (4.44).

Theorem 4.7. *Let $(M, g, e^{-f}d\mu)$ be an n-dimensional complete smooth metric measure space with $\mathrm{Ric}_f \geq 0$. Suppose u is a positive and bounded solution to the equation (4.44), where $a, b \in \mathbb{R}$.*

(i) *If $a \leq 0, b < 0$ and $u \geq e^{-2}$ then $u \equiv e^{-\frac{b}{a}}$.*

(ii) *If $a \leq 0, b \geq 0$ and $e^{-2-2\varepsilon} \leq u \leq e^{-2-\varepsilon}$ for any $\varepsilon > 0$ then u does not exist.*

Proof of Theorem 4.7. Suppose that u is a positive solution of (4.44) with $u \leq B$ for some constant $B > 0$. Since u does not depend on t, u is also a solution of the prabolic equation (4.45) in the case $a, b \in \mathbb{R}$. Furthermore, since $a \leq 0$, we have $\Gamma_a = \sup_{Q_{R,T}}\left\{(a^+)^{\frac{1}{2}} + |\nabla a|^{\frac{1}{3}}\right\} = 0$. Letting $t \to +\infty$ in (4.46) and note that $K = \Gamma_a = 0$, we get

$$\frac{|\nabla u|}{u} \leq c\left[\frac{\sqrt{A}}{R} + \sqrt{\frac{\mu^+}{R}} + \sqrt{\sup_{Q_{R,T}}\{[b(2+2\ln B - \ln u)]^+\}}\right]\sqrt{1+\ln\frac{B}{u}}. \tag{4.48}$$

(i) Since $b < 0$ and $u \geq e^{-2}$, we deduce that $A \leq 3 + \ln B$ and

$$b(2+2\ln B - \ln u) = b\left(2 + \ln u + 2\ln\frac{B}{u}\right) \leq 0.$$

This shows that $\sup_{Q_{R,T}}\{[b(2+2\ln B - \ln u)]^+\} = 0$. Then, letting $R \to +\infty$ in (4.48), we imply that $|\nabla u| = 0$. Therefore, u is a constant. Using $\triangle_f u + au +$

$bu\ln u = 0$, we get $u \equiv e^{-\frac{b}{a}}$.
(ii) Since $b \geq 0$ and $e^{-2-2\varepsilon} \leq u \leq e^{-2-\varepsilon}$ for any $\varepsilon > 0$, we obtain $A \leq 3 + 2\ln B + 2\varepsilon$, and

$$b(2 + 2\ln B - \ln u) \leq b[2 + 2(-2-\varepsilon) - (-2-2\varepsilon)] = 0.$$

This implies that $\sup_{Q_{R,T}} \{[b(2 + 2\ln B - \ln u)]^+\} = 0$. Then, letting $R \to +\infty$ in (4.46), we get $|\nabla u| = 0$. Thus, u is a constant. Using $\triangle_f u + au + bu\ln u = 0$, we deduce that $u \equiv e^{-\frac{b}{a}} \geq 1$, but $u \leq e^{-2-\varepsilon} < 1$. So, u does not exist. We complete the proof of Theorem 4.7. □

Remark 4.5. By Theorem 4.7, we see that if $e^{-2} \leq u \leq B$ is a positive solution of the nonlinear elliptic equation $\Delta_f u + bu\ln u = 0$ where $b \leq 0$ on the smooth metric measure space $(M, g, e^{-f}d\mu)$ with $\mathrm{Ric}_f \geq 0$ then $u \equiv 1$. Our Liouville type result is similar to Corollary 1.1 of Jiang [50] and Theorem 1.3 of Wu [105] in the case of $b \leq 0$.

On the other hand, from Corollary 4.5, we can give a local parabolic gradient estimate for positive smooth solutions to the following nonlinear parabolic equation

$$u_t(x,t) = \Delta u(x,t) + a(x,t)u(x,t) + bu(x,t)\ln u(x,t), \tag{4.49}$$

on Riemaniann manifolds along super Ricci flow (1.4). This is a version of the equation (4.45) when f is a constant.

Theorem 4.8. *Under the same assumption as in Theorem 4.5, if $u(x,t) \leq B$ for some constant $B > 0$, is a smooth solution to the nonlinear parabolic equation (4.45) in $\mathcal{Q}_{R,T}$, then there exists a constant c depending only n such that*

$$\frac{|\nabla u|}{u} \leq c\left[\frac{\sqrt{A}}{R} + \frac{1}{\sqrt{t}} + \sqrt{\kappa} + \sqrt{\sup_{\mathcal{Q}_{R,T}} \{[b(2 + 2\ln B - \ln u)]^+\}} + \Gamma_a\right]\sqrt{1 + \ln\frac{B}{u}} \tag{4.50}$$

in $\mathcal{Q}_{\frac{R}{2},T}$ with $t \neq 0$, where A, Γ_a are the same as Theorem 4.5.

Remark 4.6. Since $1 + \ln\frac{B}{u} \geq 1$, we get

$$\begin{aligned}\max\{b(2 + 2\ln B - \ln u), 0\} &= \max\left\{b\left(1 + \ln\frac{B}{u}\right) + b\ln B, 0\right\}\\ &\leq \left(1 + \ln\frac{B}{u}\right)\max\{b, 0\} + \max\{b\ln B, 0\}.\end{aligned}$$

Besides, we have

$$\sup_{Q_{R,T}}\left\{(a^+)^{\frac{1}{2}}+|\nabla a|^{\frac{1}{3}}\right\}\le\sup_{Q_{R,T}}\left\{(a^+)^{\frac{1}{2}}\right\}+\sup_{Q_{R,T}}\left\{|\nabla a|^{\frac{1}{3}}\right\}. \tag{4.51}$$

The above two inequalities show that our theorem is better than Theorem 1.1 of Wang in [102].

When $b=0$, from Theorem 4.8, we obtain the following result.

Corollary 4.6. *Under the same assumption as in Theorem 4.5, if $0<u(x,t)\le B$ for some constant $B>0$ is a smooth solution to the Schrödinger equation $u_t=\Delta u+au$ in $\mathcal{Q}_{R,T}$, then there exists a constant c depending only n such that*

$$\frac{|\nabla u|}{u}\le c\left[\frac{\sqrt{A}}{R}+\frac{1}{\sqrt{t}}+\sqrt{\kappa}+\sup_{\mathcal{Q}_{R,T}}\left\{(a^+)^{\frac{1}{2}}+|\nabla a|^{\frac{1}{3}}\right\}\right]\sqrt{1+\ln\frac{B}{u}} \tag{4.52}$$

in $\mathcal{Q}_{\frac{R}{2},T}$ with $t\neq 0$, where A is the same as Theorem 4.5.

Remark 4.7. By the inequality (4.51), we see that the estimate (4.52) is stronger than Corollary 1.3 of Wang [102]. Our result can be considered as a generalization along the super Ricci flow of the results of Ruan [84] and Zhu [116].

In particular, when $b=0$ and $a(x,t)\equiv 0$, by Theorem 4.8, we can derive the following local space-only gradient estimate for the linear heat equation under the super Ricci flow.

Corollary 4.7. *Under the same assumption as in Theorem 4.5, if $0<u(x,t)\le B$ for some constant $B>0$ is a smooth solution to the linear heat equation $u_t=\Delta u$ in $\mathcal{Q}_{R,T}$, then there exists a constant c depending only n such that*

$$\frac{|\nabla u|}{u}\le c\left(\frac{\sqrt{A}}{R}+\frac{1}{\sqrt{t}}+\sqrt{\kappa}\right)\sqrt{1+\ln\frac{B}{u}} \tag{4.53}$$

in $\mathcal{Q}_{\frac{R}{2},T}$ with $t\neq 0$, where A is the same as Theorem 4.5.

Remark 4.8. Since $u_t=\Delta u$, let $v=u+1$; then v satisfies $v_t=\Delta v$. Thus, without loss of generality, we may assume that $u\ge 1$. Then, we get $A=1+\ln B$ and the inequality (4.53) becomes

$$\frac{|\nabla u|}{u}\le c\left(\frac{\sqrt{1+\ln B}}{R}+\frac{1}{\sqrt{t}}+\sqrt{\kappa}\right)\sqrt{1+\ln\frac{B}{u}}.$$

Since $\sqrt{1+\ln\frac{B}{u}} \leq 1+\ln\frac{B}{u}$, our results can be seen as an improvement and extension of Theorem 1.1 of Bailesteanu-Cao-Pulemotov [8].

Applying Theorem 4.8, we obtain the new local gradient estimate for the equation (4.45) under Yamabe flow.

Corollary 4.8. *Under the same assumption as in Theorem 4.3, if* $0 < u(x,t) \leq B$ *for some constant* $B > 0$ *is a smooth solution to the equation* (4.45) *in* $\mathcal{Q}_{R,T}$, *then there exists a constant* c *depending only* n *such that*

$$\frac{|\nabla u|}{u} \leq c\left[\frac{\sqrt{A}}{R} + \sqrt{\frac{\mu^+}{R}} + \frac{1}{\sqrt{t}} + \sqrt[4]{K^2+\mathcal{H}^2} + \sqrt{\mathcal{P}_0} + \Gamma_a\right]\sqrt{1+\ln\frac{B}{u}}$$

in $\mathcal{Q}_{\frac{R}{2},T}$ *with* $t \neq 0$, *where* A, Γ_a *are the same as Theorem 4.3 and*

$$\mathcal{P}_0 = \sup_{\mathcal{Q}_{R,T}}\left\{[b(2+2\ln B - \ln u)]^+\right\}.$$

Next, we will derive a series of gradient estimates and Liouville type results for positive solutions of the following parabolic partial differential equation

$$u_t(x,t) = \Delta_f u(x,t) + a(x,t)u(x,t) + bu(x,t)(\ln u(x,t))^\alpha \tag{4.54}$$

on smooth metric measure spaces $(M, g, e^{-f}d\mu)$, where $\alpha, b \in \mathbb{R}$. This equation can be seen as a generalized version of (4.45). We first obtain the following result.

Theorem 4.9. *Under the same assumption as in Theorem 4.6, if* $1 \leq u(x,t) \leq B$ *for some constant* B, *is a smooth solution to the nonlinear parabolic equation* (4.54) *in* $Q_{R,T}$ *then there exists a constant* c *depending only* n *such that*

$$\frac{|\nabla u|}{u} \leq c\left[\frac{\sqrt{1+\ln B}}{R} + \sqrt{\frac{\mu^+}{R}} + \frac{1}{\sqrt{t}} + \sqrt{K} + \sqrt{\mathcal{P}_1} + \Gamma_a\right]\sqrt{1+\ln\frac{B}{u}}, \tag{4.55}$$

in $\mathcal{Q}_{\frac{R}{2},T}$ *with* $t \neq 0$, *where* $\Gamma_a = \sup_{Q_{R,T}}\left\{(a^+)^{\frac{1}{2}} + |\nabla a|^{\frac{1}{3}}\right\}$ *and*

$$\mathcal{P}_1 = \sup_{Q_{R,T}}\left[b\left(2\alpha + \frac{\ln u}{1+\ln B - \ln u}\right)\right]^+ \sup_{Q_{R,T}}\left\{(\ln u)^{\alpha-1}\right\}.$$

Proof of Theorem 4.9. We will apply the Theorem 4.2 to the function $F(u) =$

$bu(\ln u)^\alpha$ to prove Theorem 4.9. Observe that

$$2F'(u) - \frac{2F(u)}{u} + \frac{1}{h^2}\frac{F(u)}{u} = 2\alpha b(\ln u)^{\alpha-1} + \frac{1}{h^2}b(\ln u)^\alpha$$
$$= b\left(2\alpha + \frac{1}{h^2}\ln u\right)(\ln u)^{\alpha-1},$$

where $h = \sqrt{1+\ln\frac{B}{u}} \geq 1$. From this, we get

$$\mathcal{P} \leq \sup_{Q_{R,T}}\left[b\left(2\alpha + \frac{\ln u}{1+\ln B - \ln u}\right)\right]^+ \sup_{Q_{R,T}}\left\{(\ln u)^{\alpha-1}\right\} = \mathcal{P}_1.$$

The proof is complete. □

Remark 4.9. For $h = \sqrt{1+\ln\frac{B}{u}} \geq 1$, we see that

$$\left[b\left(2\alpha + \frac{\ln u}{1+\ln B - \ln u}\right)\right]^+ = \max\left\{b\left(2\alpha + \frac{\ln u}{h^2}\right), 0\right\}$$
$$\leq \max\{2\alpha b, 0\} + \frac{1}{h^2}\max\{b\ln u, 0\}$$
$$\leq \max\{2\alpha b, 0\} + \ln B \max\{b\ln u, 0\}.$$

Moreover, we have

$$\sqrt{1+\ln\frac{B}{u}} \leq 1 + \ln\frac{B}{u}, \quad \sqrt{1+\ln\frac{B}{u}} \leq \sqrt{\frac{B}{u}}.$$

Therefore, Theorem 4.10 is stronger than Theorem 1.3, Theorem 1.4 of Yang-Zhang [109] and Theorem 1.1 of Dung-Linh-Thu [37].

An interesting application of Theorem 4.9 is the following Liouville type result for positive solutions of nonlinear elliptic equations of the form (4.56) under the assumption $\mathrm{Ric}_f \geq 0$.

Corollary 4.9. *Let $(M, g, e^{-f}d\mu)$ be an n-dimensional complete smooth metric measure space with $\mathrm{Ric}_f \geq 0$. Assume that $1 \leq u(x,t) \leq B$ for some positive constant B, is a smooth solution to the following nonlinear elliptic equation*

$$\Delta_f u + bu(\ln u)^\alpha = 0, \tag{4.56}$$

where $\alpha, b \in \mathbb{R}$.

(i) *If $b < 0, \alpha > 0$ then $u \equiv 1$.*

(ii) *If* $b > 0, \alpha < 0$ *and* $1 \leq u \leq e^{-2\alpha}$ *then* $u \equiv 1$.

Proof of Corollary 4.9. Suppose that u is a positive solution of (4.56) with $1 \leq u \leq B$ for some constant $B > 0$. Since u does not depend on t, u is also a solution of the prabolic equation (4.54). Letting $t \to +\infty$ in (4.55) and note that $K = \Gamma_a = 0$, we get

$$\frac{|\nabla u|}{u} \leq c\left[\frac{\sqrt{1+\ln B}}{R} + \sqrt{\frac{\mu^+}{R}} + \sqrt{\mathcal{P}_1}\right]\sqrt{1+\ln\frac{B}{u}}, \tag{4.57}$$

(i) Since $b < 0, \alpha > 0$ and $u \geq 1$, we obtain $\mathcal{P}_1 = 0$. Then, letting $R \to +\infty$ in (4.57), we have $|\nabla u| = 0$. Thus, u is a constant. From (4.56), we get $u \equiv 1$.

(ii) Since $\alpha < 0, 1 \leq u \leq e^{-2\alpha}$ and $h = \sqrt{1+\ln\frac{B}{u}} \geq 1$, we deduce that

$$2\alpha + \frac{\ln u}{1+\ln B - \ln u} = 2\alpha + \ln u + \left(\frac{1}{h^2} - 1\right)\ln u \leq 2\alpha + \ln e^{-2\alpha} + \frac{1-h^2}{h^2}\ln u \leq 0.$$

From this and note that $b > 0$, we obtain $\mathcal{P}_1 = 0$. Then, letting $R \to +\infty$ in (4.57), we get $|\nabla u| = 0$. This shows that u is a constant. Using (4.56), we imply that $u \equiv 1$. □

Remark 4.10. Our Liouville type result can be seen as an extension and improvement of Corollary 3.5 in [1], Theorem 1.2 in [37], and Theorem 5.2 in [**?**].

Besides, from Theorem 4.5, we obtain a local gradient estimate for the following nonlinear parabolic equation

$$u_t(x,t) = \Delta u(x,t) + a(x,t)u(x,t) + bu(x,t)(\ln u(x,t))^\alpha \tag{4.58}$$

on Riemannian manifolds along super Ricci flow (1.4).

Corollary 4.10. *Under the same assumption as in Theorem 4.5, if* $1 \leq u(x,t) \leq B$ *for some constant* $B > 0$ *is a smooth solution to the nonlinear heat equation* (4.58) *in* $Q_{R,T}$*, then there exists a constant* c *depending only* n *such that*

$$\frac{|\nabla u|}{u} \leq c\left(\frac{\sqrt{1+\ln B}}{R} + \frac{1}{\sqrt{t}} + \sqrt{\kappa} + \sqrt{\mathcal{P}_1} + \Gamma_a\right)\sqrt{1+\ln\frac{B}{u}} \tag{4.59}$$

in $\mathcal{Q}_{\frac{R}{2},T}$ *with* $t \neq 0$, *where* $\Gamma_a = \sup_{\mathcal{Q}_{R,T}} \left\{ (a^+)^{\frac{1}{2}} + |\nabla a|^{\frac{1}{3}} \right\}$, *and*

$$\mathcal{P}_1 = \sup_{\mathcal{Q}_{R,T}} \left[b\left(2\alpha + \frac{\ln u}{1 + \ln B - \ln u} \right) \right]^+ \sup_{\mathcal{Q}_{R,T}} \left\{ (\ln u)^{\alpha - 1} \right\}. \tag{4.60}$$

Remark 4.11. Our result is stronger than Theorem 1.1, Theorem 1.2 of Yang-Zhang [110]. Moreover, Corollary 4.10 can be considered as a generalization along the super Ricci flow of the results of Dung-Linh-Thu [37].

Applying Theorem 4.3, we obtain the following local gradient estimate for the equation (4.45) under Yamabe flow.

Corollary 4.11. *Under the same assumption as in Theorem 4.3, if* $1 \leq u(x,t) \leq B$ *for some constant* $B > 0$ *is a smooth solution to the equation* (4.58) *in* $\mathcal{Q}_{R,T}$, *then there exists a constant* c *depending only* n *such that*

$$\frac{|\nabla u|}{u} \leq c \left[\frac{\sqrt{1 + \ln B}}{R} + \sqrt{\frac{\mu^+}{R}} + \frac{1}{\sqrt{t}} + \sqrt[4]{K^2 + \mathcal{H}^2} + \sqrt{\mathcal{P}_1} + \Gamma_a \right] \sqrt{1 + \ln \frac{B}{u}}$$

in $\mathcal{Q}_{\frac{R}{2},T}$ *with* $t \neq 0$, *where* $A, \Gamma_a, \mathcal{P}_1$ *are the same as Corollary 4.14.*

4.3.2 On the Einstein-scalar field Lichnerowicz type equations

The purpose of this subsection was to derive Liouville type results and gradient estimates for positive, smooth solutions of the following nonlinear elliptic equation on smooth metric measure spaces $(M, g, e^{-f}d\mu)$ of dimension $n \geq 3$,

$$\Delta_f u(x) + a(x)u(x) + bu^\alpha(x) + cu^\beta(x) = 0 \tag{4.61}$$

and its parabolic counterpart

$$u_t(x,t) = \Delta_f u(x,t) + a(x,t)u(x,t) + bu^\alpha(x,t) + cu^\beta(x,t) \tag{4.62}$$

Here, $\alpha, \beta, b, c \in \mathbb{R}$, $a(x)$ is a $\mathcal{C}^1$ function of x in (4.61), and $a(x,t)$ is a function which is $\mathcal{C}^2$ in the x-variable and $\mathcal{C}^1$ in the t-variable in (4.62).

Assume that the Bakry-Émery Ricci Ric_f is bounded below, we now apply Theorem 4.2 to derive a local gradient estimate for the equation (4.62) on the static smooth metric measure space $(M, g, e^{-f}d\mu)$. For $F(u) = bu^\alpha + cu^\beta$, we have

$$
\begin{aligned}
&2F'(u) - \frac{2F(u)}{u} + \frac{1}{h^2}\frac{F(u)}{u} \\
&= 2[(\alpha-1)b]u^{\alpha-1} + \frac{1}{h^2}bu^{\alpha-1} + [(2\beta-1)c]u^{\beta-1} + \left(\frac{1}{h^2}-1\right)cu^{\beta-1}, \qquad (4.63)
\end{aligned}
$$

where $h = \sqrt{1+\ln\frac{B}{u}} \geq 1$. Note that $0 < \frac{1}{h^2} \leq 1$ and $\frac{1}{h^2} - 1 \leq 0$. Thus, we get

$$
\begin{aligned}
&2(\alpha-1)bu^{\alpha-1} + \frac{1}{h^2}bu^{\alpha-1} \\
&\leq 2[(\alpha-1)b]^+u^{\alpha-1} + b^+u^{\alpha-1} \leq \{2[(\alpha-1)b]^+ + b^+\}\sup_{Q_{R,T}}\left\{u^{\alpha-1}\right\}, \qquad (4.64)
\end{aligned}
$$

and

$$
\begin{aligned}
&(2\beta-1)cu^{\beta-1} + \left(\frac{1}{h^2}-1\right)cu^{\beta-1} \\
&\leq [(2\beta-1)c]^+u^{\beta-1} + \frac{c^-}{h^2}u^{\beta-1} - c^-u^{\beta-1} \leq \{[(2\beta-1)c]^+ - c^-\}\sup_{Q_{R,T}}\left\{u^{\beta-1}\right\}.
\end{aligned}
$$

Plugging this and (4.64) into (4.63) , we imply that

$$
\mathcal{P} \leq \{2[(\alpha-1)b]^+ + b^+\}\sup_{Q_{R,T}}\left\{u^{\alpha-1}\right\} + \{[(2\beta-1)c]^+ - c^-\}\sup_{Q_{R,T}}\left\{u^{\beta-1}\right\} = \mathcal{P}_2.
$$

From this and Theorem 4.3, we obtain the following gradient estimate result for positive solutions of the equation (4.62).

Theorem 4.10. *Under the same assumption as in Theorem 4.2, if $0 < u(x,t) \leq B$ for some constant B, is a smooth solution to the nonlinear parabolic equation (4.62) in $Q_{R,T}$, then there exists a constant c depending only n such that*

$$
\frac{|\nabla u|}{u} \leq c\left[\frac{\sqrt{A}}{R} + \sqrt{\frac{\mu^+}{R}} + \frac{1}{\sqrt{t}} + \sqrt{K} + \mathcal{P}_2 + \Gamma_a\right]\sqrt{1+\ln\frac{B}{u}} \qquad (4.65)
$$

in $Q_{\frac{R}{2},T}$ with $t \neq 0$, where

$$
A = 1 + \ln B - \ln\left(\inf_{Q_{R,T}} u\right), \qquad \Gamma_a = \sup_{Q_{R,T}}\left\{(a^+)^{\frac{1}{2}} + |\nabla a|^{\frac{1}{3}}\right\},
$$

and

$$
\mathcal{P}_2 = \sqrt{2[(\alpha-1)b]^+ + b^+}\sup_{Q_{R,T}}\left\{u^{\frac{\alpha-1}{2}}\right\} + \sqrt{[(2\beta-1)c]^+ - c^-}\sup_{Q_{R,T}}\left\{u^{\frac{\beta-1}{2}}\right\}.
$$

As an application of Theorem 4.10, we can get a Liouville type result for positive solutions of the Einstein-scalar field Lichnerowicz type equation (4.61).

Corollary 4.12. *Let $(M, g, e^{-f}d\mu)$ be an n-dimensional complete smooth metric measure space with* $\mathrm{Ric}_f \geq 0$. *Assume that $\delta \leq u(x,t) \leq B$ for some positive constants δ and B, is a smooth solution to the equation (4.61). If a, b, c, α, β are constants satisfying $a \leq 0, b \leq 0, c \geq 0, \alpha \geq 1$ and $\beta \leq \frac{1}{2}$, then u is constant.*

Proof of Corollary 4.12. Suppose that u is a positive solution of (4.44) with $\delta \leq u \leq B$ for some constants $\delta, B > 0$. Since u does not depend on t, u is also a solution of the prabolic equation (4.62) in the case $a, b, c, \alpha, \beta \in \mathbb{R}$. Furthermore, since $a \leq 0$, we have $\Gamma_a = 0$. From the assumption $b \leq 0, c \geq 0, \alpha \geq 1$ and $\beta \leq \frac{1}{2}$, we see that

$$2[(\alpha-1)b]^+ + b^+ = 0, \quad [(2\beta-1)c]^+ - c^- = 0.$$

This shows that $\mathcal{P}_2 = 0$. Letting $t \to +\infty$ in (4.65) and note that $K = \Gamma_a = \mathcal{P}_2 = 0$, we get

$$\frac{|\nabla u|}{u} \leq c\left[\frac{\sqrt{A}}{R} + \sqrt{\frac{\mu^+}{R}}\right]\sqrt{1+\ln\frac{B}{u}}. \tag{4.66}$$

Since $\delta \leq u \leq D$, we deduce that $A = 1 + \ln B - \ln\left(\inf_{Q_{R,T}} u\right) \leq 1 + \ln B - \ln \delta$. Then, letting $R \to +\infty$ in (4.66), we obtain $\frac{|\nabla u|}{u} \leq 0$. Thus, u is a constant. We finish the proof. □

Remark 4.12. It is worth noting that Corollary 4.12 is an improvement of Dung-Khanh-Ngo's result (see [36], Corollary 2.5).

An immediate application of Corollary 4.8 is the following Liouville type result for positive solutions of Yamabe-type equations of the form (4.67) below.

Corollary 4.13. *Let $(M, g, e^{-f}d\mu)$ be an n-dimensional complete smooth metric measure space with* $\mathrm{Ric}_f \geq 0$. *Suppose that α, a, b are real numbers. Assume that $\delta \leq u(x,t) \leq B$ for some positive constants δ and B, is a smooth solution to the following equation*

$$\Delta_f u + au + bu^\alpha = 0. \tag{4.67}$$

(i) *If $\alpha \geq 1, a < 0, b < 0$, then u does not exist.*

(ii) *If $\alpha \leq \frac{1}{2}, a < 0, b > 0$, then $u = \sqrt[\alpha-1]{\frac{-a}{b}}$.*

Using Theorem 4.5, we can derive a local parabolic gradient estimate for positive smooth solutions to the following nonlinear parabolic equation

$$u_t(x,t) = \Delta u(x,t) + a(x,t)u(x,t) + bu^{\alpha}(x,t) + cu^{\beta}(x,t), \tag{4.68}$$

on Riemaniann manifolds along super Ricci flow (1.4).

Corollary 4.14. *Under the same assumption as in Theorem 4.5, if $u(x,t) \le B$ for some constant $B > 0$ in $\mathcal{Q}_{R,T}$, then there exists a constant c depending only n such that*

$$\frac{|\nabla u|}{u} \le c\left[\frac{\sqrt{A}}{R} + \frac{1}{\sqrt{t}} + \sqrt{\kappa} + \mathcal{P}_2 + \Gamma_a\right]\sqrt{1 + \ln\frac{B}{u}} \tag{4.69}$$

in $\mathcal{Q}_{\frac{R}{2},T}$ with $t \ne 0$, where

$$A = 1 + \ln B - \ln\left(\inf_{\mathcal{Q}_{R,T}} u\right), \quad \Gamma_a = \sup_{\mathcal{Q}_{R,T}}\left\{(a^+)^{\frac{1}{2}} + |\nabla a|^{\frac{1}{3}}\right\},$$

and

$$\mathcal{P}_2 = \sqrt{2[(\alpha-1)b]^+ + b^+}\sup_{\mathcal{Q}_{R,T}}\left\{u^{\frac{\alpha-1}{2}}\right\} + \sqrt{[(2\beta-1)c]^+ - c^-}\sup_{\mathcal{Q}_{R,T}}\left\{u^{\frac{\beta-1}{2}}\right\}.$$

Moreover, applying Theorem 4.8, we obtain the following local gradient estimate for the equation (4.68) under Yamabe flow.

Corollary 4.15. *Under the same assumption as in Theorem 4.3, if $0 < u(x,t) \le B$ for some constant $B > 0$ is a smooth solution to the equation (4.68) in $\mathcal{Q}_{R,T}$, then there exists a constant c depending only n such that*

$$\frac{|\nabla u|}{u} \le c\left[\frac{\sqrt{A}}{R} + \sqrt{\frac{\mu^+}{R}} + \frac{1}{\sqrt{t}} + \sqrt[4]{K^2 + \mathcal{H}^2} + \mathcal{P}_2 + \Gamma_a\right]\sqrt{1 + \ln\frac{B}{u}} \tag{4.70}$$

in $\mathcal{Q}_{\frac{R}{2},T}$ with $t \ne 0$, where $A, \mathcal{P}_2, \Gamma_a$ are the same as Corollary 4.14.

Inspired by the recent work due to Dung-Khanh-Ngo [36], in the last of this subsection, we will study the gradient estimate for solutions of the following general f-heat equation

$$u_t = \Delta_f u + au + bu\ln u + Au^{\alpha} + Bu^{\beta} \tag{4.71}$$

on complete smooth metric measure spaces $(M, g, e^{-f}d\mu)$ of dimension $n \ge 3$, where $a, b, A, B, \alpha,$ and β be constants with $A \le 0, B \ge 0, \alpha \ge 1, \beta \le \frac{1}{2}$. We first obtain the following result.

Theorem 4.11. *Let $(M, g, e^{-f}d\mu)$ be an n-dimensional complete smooth metric measure space with $\mathrm{Ric}_f \geq -(n-1)K$ for some constant $K \geq 0$ in $B(x_0, R)$. Assume that $u \in (0,1]$ is a smooth solution to the nonlinear heat equation (4.71) in $Q_{R,T}$. Then there exists a constant c depending only n such that*

$$\frac{|\nabla u|}{u} \leq c\left[\frac{\sqrt{1-\ln\left(\inf_{Q_{R,T}} u\right)}}{R} + \sqrt{\frac{\mu^+}{R}} + \frac{1}{\sqrt{t}} + \sqrt{K} + \sqrt{\Lambda}\right]\sqrt{1-\ln u},$$

for all $(x,t) \in \mathcal{Q}_{\frac{R}{2},T}$ with $t \neq 0$, where $\Lambda = \max\{b + \max\{a+b, 0\}, 0\}$.

Proof of Theorem 4.11. We will apply the Theorem 4.2 to the function

$$F(u) = au + bu\ln u + Au^\alpha + Bu^\beta$$

to prove Theorem 4.11. Observe that

$$\begin{aligned}
&2F'(u) - \frac{2F(u)}{u} + \frac{1}{h^2}\frac{F(u)}{u} \\
&= 2b + \frac{1}{h^2}(a + b\ln u) + 2[(\alpha-1)A]u^{\alpha-1} + \frac{1}{h^2}Au^{\alpha-1} \\
&\quad + [(2\beta-1)B]u^{\beta-1} + \left(\frac{1}{h^2} - 1\right)Bu^{\beta-1},
\end{aligned}$$

where $h = \sqrt{1-\ln u} \geq 1$. We notice that

$$\begin{aligned}
2b + \frac{1}{h^2}(a + b\ln u) = b + \frac{a+b}{h^2} &\leq b + \max\{a+b, 0\} \\
&\leq \max\{b + \max\{a+b, 0\}, 0\}.
\end{aligned}$$

Since $A \leq 0, B \geq 0, \alpha \geq 1, \beta \leq \frac{1}{2}$, we obtain

$$2[(\alpha-1)A]u^{\alpha-1} + \frac{1}{h^2}Au^{\alpha-1} + [(2\beta-1)B]u^{\beta-1} + \left(\frac{1}{h^2} - 1\right)Bu^{\beta-1} \leq 0.$$

From the above results, we imply that

$$\mathcal{P} \leq \max\{b + \max\{a+b, 0\}, 0\} = \Lambda.$$

The proof is complete. $\square$

When f is constant, using Theorem 4.5, we can give a local gradient estimate for the positive bounded solutions to the equation (4.71) under the super Ricci flow.

Theorem 4.12. *Let $(M, g(x,t))_{t\in[0,T]}$ be a complete solution to the super Ricci flow (1.4) and u be a smooth positive solution to the nonlinear heat equation*

$$u_t = \Delta u + au + bu\ln u + Au^{\alpha} + Bu^{\beta} \tag{4.72}$$

in the set $\mathcal{Q}_{R,T}$, where a, b, A, B, α, and β be constants with $A \le 0, B \ge 0, \alpha \ge 1, \beta \le \frac{1}{2}$. Assume that $|\mathrm{Ric}(x,t)| \le \kappa$ for some constant $\kappa \ge 0$ for all $(x,t) \in \mathcal{Q}_{R,T}$. If $u \in (0,1]$, then there exists a constant c depending only n such that

$$\frac{|\nabla u|}{u} \le c\left[\frac{\sqrt{1-\ln\left(\inf_{Q_{R,T}} u\right)}}{R} + \frac{1}{\sqrt{t}} + \sqrt{\kappa} + \sqrt{\Lambda}\right]\sqrt{1-\ln u} \tag{4.73}$$

for all $(x,t) \in \mathcal{Q}_{\frac{R}{2},T}$ with $t \ne 0$, where $\Lambda = \max\{b + \max\{a+b, 0\}, 0\}$.

Remark 4.13. In the case $b \le 0$, Theorem 4.11 is better than Theorem 1.1 in [36]. Besides, Theorem 4.12 can be seen as an extension and improvement Theorem 1.2 in [36].

Chapter 5

Rigidity and vanishing theorems for complete translating solitons

This chapter is written on the basis of the paper "Ha Tuan Dung, Nguyen Thac Dung, and Tran Quang Huy (2023), Rigidity and vanishing theorems for complete translating solitons, *Manuscripta Mathematica* Vol. 172, pp. 331-352" [34]. In this chapter, we will investigate several rigidity theorems and study the connectedness at infinity of complete translators in Euclidean spaces. This content was mentioned in Problem 1.4 in Chapter 1. Recall that a submanifold $X : M^n \to \mathbb{R}^{m+n}$ of the Euclidean space is said to be a translating soliton (abbreviated by translator) for the mean curvature flow if its mean curvature vector field H satisfies the equation

$$H = V^{\perp}, \tag{5.1}$$

for some fixed unit length constant vector V in $\mathbb{R}^{n+m}$, where $V^{\perp}$ is the normal projection of V to the normal bundle of $\mathbb{R}^{n+m}$.

Assume that the L^q-norm of the trace-free second fundamental form is finite, for some $q \in \mathbb{R}$ and using a Sobolev inequality, we first show that a translator in the Euclidean space $\mathbb{R}^{n+m}$ must be a linear subspace.

Theorem 5.1. *Let $X : M^{n\geq 3} \to \mathbb{R}^{m+n}$ be a smooth complete translating soliton in the Euclidean space $\mathbb{R}^{m+n}$. If the trace-free second fundamental form Φ of M satisfies*

$$\left(\int_M |\Phi|^n d\mu\right)^{\frac{1}{n}} < K(n,a) \quad \textit{and} \quad \int_M |\Phi|^{2a} e^{\langle V,X\rangle} d\mu < \infty,$$

where

$$1 \leq a < \frac{n + \sqrt{n^2 - 2n}}{2},$$

$$K(n,a)=\sqrt{\frac{(n-2)^2\left(a-\frac{1}{2}\right)}{D^2(n)\left[\frac{(n-2)^2\left(a-\frac{1}{2}\right)}{2n\left(na-\frac{n}{2}-a^2\right)}+(n-1)^2\right]\iota a^2}},\quad \iota=\begin{cases}2 & \textit{if } m=1\\ 4 & \textit{if } m\geq 2\end{cases},$$

and $D(n)$ is the Sobolev constant defined in Lemma 5.2, then M is a linear subspace.

The proof of this theorem relies on a Sobolev inequality on immersed submanifolds, which was first verified in [47, Theorem 2.1] and [71, Theorem 2.1]. When $a=\frac{n}{2}$, Theorem 5.1 recovers Theorem 1 in [101]. As noted in [101], the curvature condition in Theorem 5.1 is weaker than that in Theorem 7.1 in [106]. If translators are located in a halfspace, in [48, Lemma 4.2], [49, Lemma A.1], Impera and Rimoldi proved a weighted Sobolev inequality by using the bijective correspondence found by Smoczyk [91] between translators and minimal hypersurfaces in a suitable warped product. Applying Sobolev inequality, we are able to obtain the following theorem.

Theorem 5.2. *Let $X: M^{n\geq 3}\to\mathbb{R}^{n+1}$ be a smooth complete translating soliton in the Euclidean space $\mathbb{R}^{n+1}$ contained in the halfspace*

$$\Pi_{V,a}=\left\{y\in\mathbb{R}^{n+1}:\langle y,V\rangle\geq a\right\},$$

for some $a\in\mathbb{R}$. If the second fundamental form A of M satisfies

$$\left(\int_M |A|^n\varrho d\mu\right)^{\frac{1}{n}}<\sqrt{\frac{(n^2-2n+2)(n-2)^2}{n^3S(n)^2(n-1)^2}},$$

where $S(n)$ is the Sobolev constant given in Lemma 4.2 in [48] and $\rho=e^{\langle V,X\rangle}$, then M is a hyperplane.

Compared with the results in [101, Theorem 1], [106, Theorem 7.1], this result drops the assumption on the smallness of the L^n-norm of $|A|$, instead of this, we require the weighed L^n-norm of $|A|$ to be small. In fact, in [106], the author supposed that the weighted L^n-norm of $|A|$ is finite and the L^n-norm is small. Hence, when the weighted L^n-norm of $|A|$ is small, this theorem can be considered as a refinement of Theorem 7.1 in [106]. Moreover, using the weighted Sobolev inequality, we obtain a vanishing theorem as follows.

Theorem 5.3. *Let $X: M^{n\geq 3}\to\mathbb{R}^{n+1}$ be a smooth complete translating soliton* *in the Euclidean space* $\mathbb{R}^{n+1}$ *contained in the halfspace*

$$\Pi_{V,a} = \left\{ y \in \mathbb{R}^{n+1} : \langle y, V\rangle \geq a \right\},$$

for some $a \in \mathbb{R}$. *Assume that for any* $p \geq 2$,

$$\left(\int_M |A|^n e^{-f} d\mu \right)^{\frac{1}{n}} < \frac{\sqrt{(p-1)}(n-1)}{pS(n)},$$

where $f = -\langle X, V\rangle$ *and* $S(n)$ *is the Sobolev constant as in Lemma 5.4. Then there are no nontrivial* L^p_f f*-harmonic* 1*-forms on* M.

Recall that a 1-form ω is called by L^p_f f-harmonic if it satisfies

$$\triangle_f \omega = 0, \quad \int_M |\omega|^p e^{-f} d\mu < \infty.$$

Chapter 5 has three sections. Section 5.1 is used to derive some rigidity theorems. Then we prove a vanishing result for weighted harmonic forms in Section 5.2. Finally, we study translators in the Euclidean space with a Sobolev inequality in Section 5.3 and give another rigidity theorem.

5.1 Rigidity theorems

Let $X : M \to \mathbb{R}^{n+m}$ be an n-dimensional translating soliton. H, A, Φ denote the mean curvature vector, the second fundamental form, and the trace-free second fundamental form of M, respectively. Suppose that V is the unit vector such that $V^\perp = H$. Let $f = -\langle V, X\rangle$, we define

$$\Delta_f = \Delta + \langle V, \nabla(\cdot)\rangle = e^{-\langle V,X\rangle} \operatorname{div}\left(e^{\langle V,X\rangle} \nabla(\cdot) \right) = e^f \operatorname{div}(e^{-f} \nabla(\cdot)).$$

The trace-free second fundamental form is given by $\Phi = A - \frac{1}{n} g \otimes H$. It is well known that

$$|\Phi|^2 = |A|^2 - \frac{1}{n}|H|^2 \quad \text{and} \quad |\nabla\Phi|^2 = |\nabla A|^2 - \frac{1}{n}\nabla|H|^2.$$

In order to prove our theorems, we need the following Simons type identity, which has been obtained by Xin [101, Lemma 3] (see also [106, Proposition 2.1]).

Lemma 5.1. [101, Lemma 3] *On a translating soliton* M^n *in* $\mathbb{R}^{n+m}$, *we have*

$$\Delta_f |\Phi|^2 \geq 2|\nabla|\Phi||^2 - \iota|\Phi|^4 - \frac{2}{n}|H|^2|\Phi|^2, \tag{5.2}$$

where

$$\iota = \begin{cases} 2, & \text{if} \quad m = 1, \\ 4, & \text{if} \quad m \geq 2. \end{cases}$$

Moreover, when $m = 1$, we have

$$\Delta_f |\Phi|^2 = 2|\nabla\Phi|^2 - 2|A|^2|\Phi|^2. \tag{5.3}$$

We now recall that the following Sobolev inequality for submanifolds in the Euclidean is beneficial in deriving our rigidity theorems (see [107, Lemma 2.5]).

Lemma 5.2 (Sobolev inequality)**.** *Let $M^n\,(n \geq 3)$ be a complete submanifold in the Euclidean space $\mathbb{R}^{n+m}$. Let f be a nonnegative $\mathcal{C}^\infty$ function with compact support. Then for all $s \in \mathbb{R}^+$, we have*

$$\|f\|^2_{\frac{2n}{n-2}} \leq D^2(n) \left[\frac{4(n-1)^2(1+s)}{(n-2)^2} \|\nabla f\|_2^2 + \left(1 + \frac{1}{s}\right) \frac{1}{n^2} \||H|f\|_2^2 \right],$$

where $D(n) = 2^n(1+n)^{\frac{n+1}{n}}(n-1)^{-1}\sigma_n^{-\frac{1}{n}}$, and σ_n denotes the volume of the unit ball in $\mathbb{R}^n$.

For convenience in proving Theorem 5.1, we denote $\varrho = e^{\langle V, X\rangle}$ and $d\mu$ might be omitted in the integrations. We can follow the proof of Theorem 1 in [101], but instead of using the function f defined in Lemma 5 of [101], we use the function $\varphi = |\Phi|^a \varrho^{\frac{1}{2}} \eta$, where $a \geq 1$ is a constant to be determined later and η is a smooth function with compact support on M. For the convenience of the reader, in order to help him/her check the influence of the constant a in every step, we give all the details of the computations.

Lemma 5.3. *Assume that $|\Phi| \neq 0$ on M. If η is a smooth function with compact support on M, then*

$$\int_M |\nabla\varphi|^2 = \int_M |\nabla(|\Phi|^a\eta)|^2 \varrho - \frac{1}{2}\int_M |\Phi|^{2a}\eta^2\varrho + \frac{1}{4}\int_M |\Phi|^{2a}\left|V^\top\right|^2\eta^2\varrho. \tag{5.4}$$

Proof of Lemma 5.4. Integrating by parts, we deduce that

$$\begin{aligned} \int_M |\nabla\varphi|^2 &= \int_M |\nabla(|\Phi|^a\eta)|^2 \varrho + \frac{1}{2}\int_M \left\langle \nabla\left(|\Phi|^{2a}\eta^2\right), \nabla\varrho \right\rangle + \int_M |\Phi|^{2a}\eta^2 \left|\nabla\varrho^{\frac{1}{2}}\right|^2 \\ &= \int_M |\nabla(|\Phi|^a\eta)|^2 \varrho - \frac{1}{2}\int_M |\Phi|^{2a}\eta^2\Delta\varrho + \int_M |\Phi|^{2a}\eta^2 \left|\nabla\varrho^{\frac{1}{2}}\right|^2. \end{aligned}$$

Since M^n is a translating soliton, we have

$$\nabla\varrho = \nabla e^{\langle V,X\rangle} = \varrho V^\top, \quad \nabla \varrho^{\frac{1}{2}} = \frac{1}{2}\varrho^{-\frac{1}{2}}\nabla\varrho = \frac{1}{2}\varrho^{\frac{1}{2}}V^\top,$$

and

$$\Delta\varrho = \sum_i \nabla_i\varrho\,\langle V, e_i\rangle + \sum_i \varrho\,\langle V, \nabla_i e_i\rangle = \varrho\left(\left|V^\top\right|^2 + \left|V^\perp\right|^2\right) = \varrho.$$

Using this, we conclude that

$$\int_M |\nabla\varphi|^2 = \int_M \left|\nabla\left(|\Phi|^a\eta\right)\right|^2\varrho - \frac{1}{2}\int_M |\Phi|^{2a}\eta^2\varrho + \frac{1}{4}\int_M |\Phi|^{2a}\left|V^\top\right|^2\eta^2\varrho.$$

The proof is complete. $\square$

Now, combining the Sobolev inequality in Lemma 5.2 and (5.4), we get

$$\begin{aligned}
&\left(\int_M |\varphi|^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}} \\
&\leq D^2(n)\cdot\left\{\frac{4(n-1)^2(1+s)}{(n-2)^2}\int_M|\nabla\varphi|^2 + \left(1+\frac{1}{s}\right)\cdot\frac{1}{n^2}\int_M|H|^2\varphi^2\right\} \\
&= D^2(n)\cdot\left\{\frac{4(n-1)^2(1+s)}{(n-2)^2}\left(\int_M\left|\nabla\left(|\Phi|^a\eta\right)\right|^2\varrho - \frac{1}{2}\int_M|\Phi|^{2a}\eta^2\varrho\right.\right. \\
&\quad\left.\left. +\frac{1}{4}\int_M|\Phi|^{2a}\left|V^\top\right|^2\eta^2\varrho\right) + \left(1+\frac{1}{s}\right)\cdot\frac{1}{n^2}\int_M|\Phi|^{2a}|H|^2\eta^2\varrho\right\}.
\end{aligned}$$

Note that

$$|V^\top|^2 + |V^\perp|^2 = |V^\top|^2 + |H|^2 = 1.$$

Thus, we obtain

$$\begin{aligned}
&\left(\int_M |\varphi|^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}} \\
&\leq D^2(n)\cdot\left\{\frac{4(n-1)^2(1+s)}{(n-2)^2}\left(\int_M\left|\nabla\left(|\Phi|^a\eta\right)\right|^2\varrho - \frac{1}{4}\int_M|\Phi|^{2a}\left|V^\top\right|^2\eta^2\varrho\right.\right. \\
&\quad\left.\left. -\frac{1}{2}\int_M|\Phi|^{2a}|H|^2\eta^2\varrho\right) + \left(1+\frac{1}{s}\right)\cdot\frac{1}{n^2}\int_M|\Phi|^{2a}|H|^2\eta^2\varrho\right\} \\
&= D^2(n)\cdot\left\{\frac{4(n-1)^2(1+s)}{(n-2)^2}\left(\int_M a^2|\nabla|\Phi||^2|\Phi|^{2a-2}\eta^2\varrho\right.\right. \\
&\quad +\int_M 2a|\Phi|^{2a-1}\eta\nabla|\Phi|\cdot\nabla\eta\varrho + \int_M|\Phi|^{2a}|\nabla\eta|^2\varrho - \frac{1}{4}\int_M|\Phi|^{2a}\left|V^\top\right|^2\eta^2\varrho \\
&\quad\left.\left. -\frac{1}{2}\int_M|\Phi|^{2a}|H|^2\eta^2\varrho\right) + \left(1+\frac{1}{s}\right)\cdot\frac{1}{n^2}\int_M|\Phi|^{2a}|H|^2\eta^2\varrho\right\}.
\end{aligned} \tag{5.5}$$

By the Cauchy inequality, for $\delta > 0$ we have

$$
\begin{aligned}
&\left(\int_M |\varphi|^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}} \\
&\le \frac{4D^2(n)(n-1)^2(1+s)}{(n-2)^2}\Bigg\{(1+\delta)a^2\int_M |\nabla|\Phi||^2|\Phi|^{2a-2}\eta^2\varrho \\
&\quad + \left(1+\frac{1}{\delta}\right)\int_M |\Phi|^{2a}|\nabla\eta|^2\varrho - \frac{1}{4}\int_M |\Phi|^{2a}\left|V^\top\right|^2\eta^2\varrho - \frac{1}{2}\int_M |\Phi|^{2a}|H|^2\eta^2\varrho\Bigg\} \\
&\quad + D^2(n)\left(1+\frac{1}{s}\right)\cdot\frac{1}{n^2}\int_M |\Phi|^{2a}|H|^2\eta^2\varrho. \qquad (5.6)
\end{aligned}
$$

In order to estimate the term $\int_M |\nabla|\Phi||^2|\Phi|^{2a-2}\eta^2\varrho$, we multiply $|\Phi|^{2a-2}\eta^2$ on both sides of (5.2) and integrating by parts with respect to the measure $\varrho d\mu$ on M gives

$$
\begin{aligned}
0 \ge\ & 2\int_M |\nabla|\Phi||^2|\Phi|^{2a-2}\eta^2\varrho - \iota\int_M |\Phi|^{2a+2}\eta^2\varrho - \frac{2}{n}\int_M |\Phi|^{2a}|H|^2\eta^2\varrho \\
& - \int_M |\Phi|^{2a-2}\eta^2\Delta_f|\Phi|^2\varrho. \qquad (5.7)
\end{aligned}
$$

Since η has compact support on M, by the Stokes theorem, we obtain

$$
\begin{aligned}
& -\int_M |\Phi|^{2a-2}\eta^2\Delta_f|\Phi|^2\varrho \\
&= -\int_M |\Phi|^{2a-2}\eta^2\,\mathrm{div}\left(\varrho\,\nabla|\Phi|^2\right) \\
&= 2\int_M \varrho|\Phi|\langle\nabla|\Phi|, \nabla\left(|\Phi|^{2a-2}\eta^2\right)\rangle \\
&= 4(a-1)\int_M |\nabla|\Phi||^2|\Phi|^{2a-2}\eta^2\varrho + 4\int_M \langle\nabla|\Phi|, \nabla\eta\rangle|\Phi|^{2a-1}\eta\varrho. \qquad (5.8)
\end{aligned}
$$

Combining (5.7) and (5.8), we get

$$
\begin{aligned}
0 \ge\ & 4\left(a-\frac{1}{2}\right)\int_M |\nabla|\Phi||^2|\Phi|^{2a-2}\eta^2\varrho - \iota\int_M |\Phi|^{2a+2}\eta^2\varrho - \frac{2}{n}\int_M |\Phi|^{2a}|H|^2\eta^2\varrho \\
& + 4\int_M \langle\nabla|\Phi|, \nabla\eta\rangle|\Phi|^{2a-1}\eta\varrho.
\end{aligned}
$$

By the Cauchy inequality, for $0 < \varepsilon < a - \frac{1}{2}$, we have

$$
\begin{aligned}
&\iota\int_M |\Phi|^{2a+2}\eta^2\varrho + \frac{2}{n}\int_M |\Phi|^{2a}|H|^2\eta^2\varrho + \frac{1}{\varepsilon}\int_M |\Phi|^{2a}|\nabla\eta|^2\varrho \\
&\qquad \ge 4\left(a-\frac{1}{2}-\varepsilon\right)\int_M |\nabla|\Phi||^2|\Phi|^{2a-2}\eta^2\varrho. \qquad (5.9)
\end{aligned}
$$

Substituting (5.9) into (5.6), we get

$$\left(\int_M |\varphi|^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}} \leq \frac{4D^2(n)(n-1)^2(1+s)}{(n-2)^2}\left\{\frac{a^2(1+\delta)}{4\left(a-\frac{1}{2}-\varepsilon\right)}\left(\iota\int_M |\Phi|^{2a+2}\eta^2\varrho\right.\right.$$
$$\left.+\frac{2}{n}\int_M |\Phi|^{2a}|H|^2\eta^2\varrho+\frac{1}{\varepsilon}\int_M |\Phi|^{2a}|\nabla\eta|^2\varrho\right)$$
$$\left.+\left(1+\frac{1}{\delta}\right)\int_M |\Phi|^{2a}|\nabla\eta|^2\varrho-\frac{1}{2}\int_M |\Phi|^{2a}|H|^2\eta^2\varrho\right\}$$
$$+D^2(n)\left(1+\frac{1}{s}\right)\cdot\frac{1}{n^2}\int_M |\Phi|^{2a}|H|^2\eta^2\varrho. \tag{5.10}$$

We want to get rid of the term $\int_M |\Phi|^{2a}|H|^2\eta^2\varrho$ by choosing $\delta > 0$ appropriately. Put

$$\delta=\delta(n,\varepsilon,a)=\frac{\left(2(n-1)^2n^2s-(n-2)^2\right)\left(a-\frac{1}{2}-\varepsilon\right)}{2(n-1)^2a^2ns}-1.$$

We would require $\delta > 0$, this occurs only if s satisfies

$$s>\frac{(n-2)^2\left(a-\frac{1}{2}-\varepsilon\right)}{2(n-1)^2n\left(na-\frac{n}{2}-a^2-n\epsilon\right)} \tag{5.11}$$

for some $\varepsilon \in \left(0, a-\frac{1}{2}-\frac{a^2}{n}\right)$ defined later and also, we need $1 \leq a < \frac{n+\sqrt{n^2-2n}}{2}$. Consequently, we have

$$\kappa^{-1}\left(\int_M |\varphi|^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}} \leq \frac{a^2(1+s)(1+\delta)}{4\left(a-\frac{1}{2}-\varepsilon\right)}\left(\iota\int_M |\Phi|^{2a+2}\eta^2\varrho+\frac{1}{\varepsilon}\int_M |\Phi|^{2a}|\nabla\eta|^2\varrho\right)$$
$$+(1+s)\left(1+\frac{1}{\delta}\right)\int_M |\Phi|^{2a}|\nabla\eta|^2\varrho$$
$$=\frac{(1+s)\iota\left[2sn^2(n-1)^2-(n-2)^2\right]}{8sn(n-1)^2}\int_M |\Phi|^{2a+2}\eta^2\varrho$$
$$+C(s,\varepsilon,n,a)\int_M |\Phi|^{2a}|\nabla\eta|^2\varrho,$$

where $C(s,\varepsilon,n,a)$ is an explicit positive constant depending on s,ε,n,a and $\kappa=\frac{4D^2(n)(n-1)^2}{(n-2)^2}$. By Hölder inequality we have

$$\int_M |\Phi|^{2a+2}\eta^2\varrho \leq \left(\int_M (|\Phi|^{2\cdot\frac{n}{2}}\right)^{\frac{2}{n}}\cdot\left(\int_M \left(|\Phi|^{2a}\eta^2\varrho\right)^{\frac{n}{n-2}}\right)^{\frac{n-2}{n}}$$
$$=\left(\int_M |\Phi|^{2\cdot\frac{n}{2}}\right)^{\frac{2}{n}}\cdot\left(\int_M |\varphi|^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}}. \tag{5.12}$$

Applying this to (5.12), we get

$$\begin{aligned}
&\kappa^{-1}\left(\int_M |\varphi|^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}} \\
&\leq \frac{(1+s)\iota\left[2sn^2(n-1)^2-(n-2)^2\right]}{8sn(n-1)^2}\left(\int_M |\Phi|^{2a}\right)^{\frac{2}{n}}\cdot\left(\int_M |\varphi|^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}} \\
&+C(s,\varepsilon,n,a)\int_M |\Phi|^{2a}|\nabla\eta|^2\varrho.
\end{aligned} \tag{5.13}$$

Put

$$K(n,s)=\sqrt{\frac{8sn(n-1)^2}{(1+s)\iota\left[2sn^2(n-1)^2-(n-2)^2\right]\kappa}}.$$

By condition (5.11) we can choose

$$s=\frac{(n-2)^2\left(a-\frac{1}{2}\right)}{2(n-1)^2 n\left(na-\frac{n}{2}-a^2-n\varepsilon\right)}.$$

Hence, substituting s into $K(n,s)$, we have

$$K(n,a,\varepsilon)=K(n,s(a,\varepsilon))=\sqrt{\frac{(n-2)^2\left(a-\frac{1}{2}\right)}{D^2(n)\left[\frac{(n-2)^2\left(a-\frac{1}{2}\right)}{2n\left(na-\frac{n}{2}-a^2-n\varepsilon\right)}+(n-1)^2\right]\iota(n\varepsilon+a^2)}}. \tag{5.14}$$

Set

$$K(n,a)=\sup_{\varepsilon\in\left(0,\frac{(n-a-1)(a-1)}{n}\right)}K(n,a,\varepsilon)=\sqrt{\frac{(n-2)^2\left(a-\frac{1}{2}\right)}{D^2(n)\left[\frac{(n-2)^2\left(a-\frac{1}{2}\right)}{2n\left(na-\frac{n}{2}-a^2\right)}+(n-1)^2\right]\iota a^2}}.$$

We now can give a proof of Theorem 5.1.

Proof of Theorem 5.1. Since we made the assumption $\left(\int_M |\Phi|^n d\mu\right)^{\frac{1}{n}} < K(n,a)$, there exists a positive constant $\check{K}$ such that

$$\left(\int_M |\Phi|^n d\mu\right)^{\frac{1}{n}} < \check{K} < K(n,a). \tag{5.15}$$

Thus, there exists $\varepsilon=\varepsilon_0>0$ such that

$$\check{K} < K(n,a,\varepsilon_0) < K(n,a).$$

Using this and combining (5.13), (5.15), there exists $0 < \epsilon < 1$ such that

$$\begin{aligned}
&\kappa^{-1}\left(\int_M |f|^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}} \\
&\leq \kappa^{-1}\cdot K\left(n,a,\varepsilon_0\right)^{-2}\cdot \check{K}^2\left(\int_M |f|^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}} + \bar{C}\left(n,a,\varepsilon_0\right)\int_M |\Phi|^{2a}|\nabla\eta|^2\varrho \\
&\leq \frac{1-\epsilon}{\kappa}\left(\int_M |f|^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}} + \bar{C}\left(n,a,\varepsilon_0\right)\int_M |\Phi|^{2a}|\nabla\eta|^2\varrho
\end{aligned} \tag{5.16}$$

or equivalently

$$\frac{\epsilon}{\kappa}\left(\int_M |f|^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}} \leq \bar{C}\left(n,a,\varepsilon_0\right)\int_M |\Phi|^{2a}|\nabla\eta|^2\varrho. \tag{5.17}$$

Let $\eta(X) = \eta_r(X) = \phi\left(\frac{|X|}{r}\right)$ for any $r > 0$, where ϕ is a non-negative smooth function on $[0,+\infty)$ satisfying

$$\phi(x) = \begin{cases} 1, & \text{if} \quad x \in [0,1), \\ 0, & \text{if} \quad x \in [2,+\infty), \end{cases} \tag{5.18}$$

and $|\phi'| \leq C$ for some absolute constant. Since $\int_M |\Phi|^q\varrho$ and $\bar{C}(n,a,\varepsilon_0)$ are bounded the right-hand side of (5.17) approaches zero as $r \to \infty$, which implies the left-hand side to be equal to zero i.e. $|\Phi| \equiv 0$.

Finally, using the assertion that $|\Phi| = 0$, it was confirmed in [101, Theorem 1] that M is a linear subspace. In the rest of the proof, we give a detail of argument, which is inspired by Impera and Rimoldi in [48, Theorem A]. We argue as follows. Since $|\Phi| = 0$, it turns out that $|A|^2 = \frac{1}{n}H^2$. Moreover, we note that $|\nabla\Phi|^2 = |\nabla|\Phi||^2 = 0$. This implies

$$0 = |\nabla\Phi|^2 = |\nabla A|^2 - \frac{1}{n}|\nabla H|^2.$$

Therefore, we get

$$\left|\nabla|A|^2\right| = \frac{1}{n}\left|\nabla H^2\right| = \frac{2}{n}\left|H\right|\left|\nabla H\right| = \frac{2}{n}\left(\sqrt{n}\left|A\right|\right)\left(\sqrt{n}\left|\nabla A\right|\right) = 2\left|A\right|\left|\nabla A\right|.$$

As a consequence, $|\nabla A| = |\nabla|A||$. Therefore, we can apply the argument in the proof of Theorem A in [48] to conclude that M is a linear subspace. The proof is complete. □

Observe that $[1, n-1] \subset \left[1, \frac{n+\sqrt{n^2-2n}}{2}\right)$, the weighted L^{2a} norm of $|\Phi|$ in our theorem is wider than those in [101]. Moreover, when $a = \frac{n}{2}$, our theorem recovers the following rigidity property, which was obtained by Wang, Xu, and Zhao in [101].

Theorem 5.4. [101, Theorem 1] *Let $X : M^{n\geq 3} \to \mathbb{R}^{m+n}$ be a smooth complete translating soliton in the Euclidean space $\mathbb{R}^{m+n}$. If the trace-free second fundamental form Φ of M satisfies*

$$\left(\int_M |\Phi|^n d\mu\right)^{\frac{1}{n}} < K(n) \quad \text{and} \quad \int_M |\Phi|^n e^{\langle V,X\rangle} d\mu < \infty,$$

where $K(n)$ is defined as above, then M is a linear subspace.

It is worth mentioning that the above condition is weaker than that in the rigidity theorem of Xin [106, Theorem 7.1]. To derive another rigidity result, we can use the following version of the Sobolev inequality.

Lemma 5.4. [48, Lemma 4.2] *Let $X : M^{n\geq 3} \to \mathbb{R}^{n+1}$ be a translator contained in the halfspace $\Pi_{V,a} = \{p \in \mathbb{R}^{n+1} : \langle p, V\rangle \geq a\}$ for some $a \in \mathbb{R}$. Let u be a non-negative compactly supported $\mathcal{C}^\infty$ function on M. Then,*

$$\left[\int_M u^{\frac{2n}{n-2}} \varrho d\mu\right]^{\frac{n-2}{n}} \leq \left(\frac{2(n-1)S(n)}{n-2}\right)^2 \int_M |\nabla u|^2 \varrho d\mu, \tag{5.19}$$

where $\rho = e^{\langle V,X\rangle}$ and $S(n)$ is the Sobolev constant given in Lemma 4.2 in [48].

Repeating the same computation as above, we can give a verification of Theorem 5.2 as follows.

Proof of Theorem 5.2. Applying the Sobolev inequalities (5.19) to $u = |\Phi|^{\frac{n}{2}}\eta$ and using the Cauchy inequality, we have

$$\begin{aligned}
\left[\int_M \left(|\Phi|^{\frac{n}{2}}\eta\right)^{\frac{2n}{n-2}} \varrho d\mu\right]^{\frac{n-2}{n}} &\leq \left(\frac{2S(n)(n-1)}{n-1}\right)^2 \int_M \left|\nabla\left(|\Phi|^{\frac{n}{2}}\eta\right)\right|^2 \varrho d\mu \\
&= \left(\frac{2S(n)(n-1)}{n-2}\right)^2 \left(\int_M \frac{n^2}{4}|\nabla|\Phi||^2|\Phi|^{n-2}\eta^2 \varrho d\mu\right. \\
&\quad \left. + \int_M n|\Phi|^{n-1}\eta\langle\nabla|\Phi|, \nabla\eta\rangle \varrho d\mu + \int_M |\Phi|^n|\nabla\eta|^2 \varrho d\mu\right) \\
&\leq \left(\frac{2S(n)(n-1)}{n-2}\right)^2 \left((1+\delta)\int_M \frac{n^2}{4}|\nabla|\Phi||^2|\Phi|^{n-2}\eta^2 \varrho d\mu\right. \\
&\quad \left. + \left(1+\frac{1}{\delta}\right)\int_M |\Phi|^n|\nabla\eta|^2 \varrho d\mu\right).
\end{aligned}$$

Applying (5.9) and notice that $\iota = 2$, we have, for $0 < \varepsilon < \frac{n}{2} - \frac{1}{2}$,

$$\begin{aligned}\kappa_2^{-1}\left[\int_M \left(|\Phi|^{\frac{n}{2}}\eta\right)^{\frac{2(n)}{n-2}} \varrho d\mu\right]^{\frac{n-2}{n}} \leq & \left\{\frac{\frac{n^2}{4}(1+\delta)}{4\left(\frac{n}{2}-\frac{1}{2}-\varepsilon\right)}\left(2\int_M |\Phi|^{n+2}\eta^2 \varrho d\mu\right.\right. \\ & \left. + \frac{2}{n}\int_M |\Phi|^n|H|^2\eta^2\varrho d\mu + \frac{1}{\varepsilon}\int_M |\Phi|^n|\nabla\eta|^2\varrho d\mu\right) \\ & \left. + \left(1+\frac{1}{\delta}\right)\int_M |\Phi|^n|\nabla\eta|^2\varrho d\mu\right\},\end{aligned} \tag{5.20}$$

where $\kappa_2 = \left(\frac{2S(n)(n-1)}{n-2}\right)^2$. By the fact that

$$|A|^2 = |\Phi|^2 + \frac{1}{n}|H|^2,$$

we can rewrite (5.20) as

$$\begin{aligned}\kappa_2^{-1}\left[\int_M \left(|\Phi|^{\frac{n}{2}}\eta\right)^{\frac{2n}{n-2}} \varrho d\mu\right]^{\frac{n-2}{n}} \leq & \frac{\frac{n^2}{4}(1+\delta)}{2\left(\frac{n}{2}-\frac{1}{2}-\varepsilon\right)}\int_M |\Phi|^n|A|^2\eta^2\varrho d\mu \\ & + \widehat{C}(n,\delta,\varepsilon)\int_M |\Phi|^n|\nabla\eta|^2\varrho d\mu,\end{aligned}$$

where $\widehat{C}(n,\delta,\varepsilon)$ is explicit positive constant depending on n, δ, ε. Applying Hölder inequality, we have

$$\begin{aligned}&\kappa_2^{-1}\left[\int_M \left(|\Phi|^{\frac{n}{2}}\eta\right)^{\frac{2n}{n-2}} \varrho d\mu\right]^{\frac{n-2}{n}} \\ \leq & \frac{n^2(1+\delta)}{8\left(\frac{n}{2}-\frac{1}{2}-\varepsilon\right)}\left(\int_M (|A|^2\varrho^{\frac{2}{n}})^{\frac{n}{2}} d\mu\right)^{\frac{2}{n}} \cdot \left(\int_M \left(|\Phi|^{\frac{n}{2}}\eta\varrho^{\frac{n-2}{2n}}\right)^{\frac{2n}{n-2}} d\mu\right)^{\frac{n-2}{n}} \\ & + \widehat{C}(n,\delta,\varepsilon)\int_M |\Phi|^n|\nabla\eta|^2\varrho d\mu, \\ \leq & \frac{n^2(1+\delta)}{8\left(\frac{n}{2}-\frac{1}{2}-\varepsilon\right)}\left(\int_M |A|^n\varrho d\mu\right)^{\frac{2}{n}} \cdot \left(\int_M \left(|\Phi|^{\frac{n}{2}}\eta\right)^{\frac{2n}{n-2}} \varrho d\mu\right)^{\frac{n-2}{n}} \\ & + \widehat{C}(n,\delta,\varepsilon)\int_M |A|^n|\nabla\eta|^2\varrho d\mu,\end{aligned}$$

here we used $|\Phi| \leq |A|$ in the last inequality. Put

$$K_2(n,\varepsilon,\delta) = \sqrt{\frac{8\left(\frac{n}{2}-\frac{1}{2}-\varepsilon\right)}{n^2(1+\delta)\kappa_2}},$$

and

$$K_2(n) = \sup_{\delta>0,\, 0<\varepsilon<a-\frac{n-1}{n}} K_2(n,\varepsilon,\delta) = \sqrt{\frac{(n-2)^2}{S(n)^2(n-1)n^2}}.$$

By the assumption

$$\left(\int_M |A|^n \varrho d\mu\right)^{\frac{1}{n}} < K_2(n)$$

and using the same argument as Theorem 5.1, we complete the proof. □

Now, as mentioned in [48], an application of the maximum principle and the weighted version of a result in [29] give that translator with mean curvature that does not change sign are either f-stable (generalizing, in particular, Theorem 1.2.5 in [88], and Theorem 2.5 in [89]) or they split as the product of a line parallel to the translating direction and a minimal hypersurface in the orthogonal complement of the line. Note that, in this latter case, by Fubini theorem, the condition $|A| \in L^p(M_f)$ for some $p > 0$ is met if and only if $|A| \equiv 0$ (i.e. M is a translator hyperplane), here $M_f = (M, g, e^{-f}d\mu)$. Moreover, to adapt the ideas in [86] for minimal surface, Ma and Miquel proved in [68, Lemma 9] a refined Kato inequality on translating solitons as follows.

Lemma 5.5. [68, Lemma 9] *Let M^n be a hypersurface immersed in $\mathbb{R}^{n+1}$ satisfying*

$$|\nabla A| \le \frac{n+1}{2n}|\nabla H|,$$

then we have

$$|\nabla \Phi|^2 \ge \frac{n+1}{n}|\nabla|\Phi||^2.$$

Note that on the translating soliton M, we have $\nabla H = \langle \nabla\nu, v\rangle = A(\cdot, v)$, so the condition becomes

$$|\nabla A| \le \frac{n+1}{2n}|A(\cdot, v)|.$$

Now, under these assumptions, we obtain the following result, which can be considered as an improvement of Theorem 6 in [68].

Theorem 5.5. *Let $X : M^{n\ge 2} \to \mathbb{R}^{n+1}$ be a translator with mean curvature which does not change sign. Suppose that $|\nabla A| \le \frac{n+1}{2n}|\nabla H|$ and the traceless second fundamental form of the immersion satisfies $|\Phi| \in L^p(M_f)$ for $p \in \left(2 - \frac{2}{\sqrt{n}}, 2 + \frac{2}{\sqrt{n}}\right)$. Then M is a hyperplane.*

Proof of Theorem 5.5. Since the curvature does not change sign, we may assume that M is f-stable. Otherwise, $|A| \equiv 0$, so M is a hyperplane. From the definition of the f-Laplacian operator and the equation (5.3), we have

$$|\Phi|\,\Delta_f\,|\Phi| = |\nabla\Phi|^2 - |\nabla\,|\Phi||^2 - |A|^2|\Phi|^2.$$

By the Kato-type inequality in Lemma 5.5, this implies

$$|\Phi|\,\Delta_f\,|\Phi| \geq \frac{1}{n}|\nabla\,|\Phi||^2 - |A|^2|\Phi|^2. \tag{5.21}$$

Now, let η be a smooth compactly supported function on M. For any $a > 1$, multiplying $|\Phi|^{a-1}\eta^2$ both sides of the (5.21) and integrating by parts with respect to the measure $e^{-f}d\mu$ on M yield

$$\begin{aligned}&\int_M \eta^2\,|\Phi|^a\Delta_f\,|\Phi|\,e^{-f}d\mu\\ &\qquad\geq \frac{1}{n}\int_M \eta^2\,|\Phi|^{a-1}|\nabla\,|\Phi||^2 e^{-f}d\mu - \int_M |A|^2\eta^2|\Phi|^{a+1}e^{-f}d\mu.\end{aligned} \tag{5.22}$$

Since η has compact support on M, by the Stokes theorem, it shows that

$$\begin{aligned}&\int_M \eta^2|\Phi|^a\Delta_f\,|\Phi|\,e^{-f}d\mu\\ &= -\int_M \left\langle\nabla\left(\eta^2|\Phi|^a\right),\nabla\,|\Phi|\right\rangle e^{-f}d\mu\\ &= -\int_M \left\langle 2\eta|\Phi|^a\nabla\eta + a\eta^2|\Phi|^{a-1}\nabla\,|\Phi|\,,\nabla\,|\Phi|\right\rangle e^{-f}d\mu\\ &= -2\int_M |\Phi|^a\left\langle\nabla\eta,\nabla\,|\Phi|\right\rangle\eta e^{-f}d\mu - a\int_M \eta^2|\Phi|^{a-1}|\nabla\,|\Phi||^2 e^{-f}d\mu.\end{aligned}$$

Substituting the above identity into (5.22), we obtain

$$\begin{aligned}&-2\int_M |\Phi|^a\left\langle\nabla\eta,\nabla\,|\Phi|\right\rangle\eta e^{-f}d\mu - a\int_M \eta^2|\Phi|^{a-1}|\nabla\,|\Phi||^2 e^{-f}d\mu\\ &\qquad\geq \frac{1}{n}\int_M \eta^2\,|\Phi|^{a-1}|\nabla\,|\Phi||^2 e^{-f}d\mu - \int_M |A|^2\eta^2|\Phi|^{a+1}e^{-f}d\mu,\end{aligned}$$

or equivalently

$$\begin{aligned}&\left(a+\frac{1}{n}\right)\int_M \eta^2|\Phi|^{a-1}|\nabla\,|\Phi||^2 e^{-f}d\mu\\ &\qquad\leq 2\int_M |\Phi|^a\left\langle\nabla\eta,\nabla\,|\Phi|\right\rangle\eta e^{-f}d\mu + \int_M |A|^2\eta^2|\Phi|^{a+1}e^{-f}d\mu.\end{aligned} \tag{5.23}$$

On the other hand, since M satisfies the stability inequality, we have

$$\int_M |A|^2\psi^2 e^{-f}d\mu \leq \int_M |\nabla\psi|^2 e^{-f}d\mu.$$

Replacing ψ by $\eta|\Phi|^{\frac{a+1}{2}}$ in the above inequality gives

$$\begin{aligned}
&\int_M |A|^2\eta^2|\Phi|^{a+1}e^{-f}d\mu \\
&\leq \int_M \left|\nabla\left(\eta|\Phi|^{\frac{a+1}{2}}\right)\right|^2 e^{-f}d\mu \\
&= \int_M |\Phi|^{a+1}|\nabla\eta|^2 e^{-f}d\mu + (a+1)\int_M |\Phi|^a\left\langle \nabla\eta, \nabla|\Phi|\right\rangle \eta e^{-f}d\mu \\
&\quad + \frac{(a+1)^2}{4}\int_M |\Phi|^{a-1}|\nabla|\nabla\Phi||^2\eta^2 e^{-f}d\mu. \qquad (5.24)
\end{aligned}$$

Combining (5.23) and (5.24), we have

$$\begin{aligned}
&\left(a+\frac{1}{n}\right)\int_M \eta^2|\Phi|^{a-1}|\nabla|\Phi||^2 e^{-f}d\mu \\
&\leq 2\int_M |\Phi|^a\left\langle\nabla\eta,\nabla|\Phi|\right\rangle\eta e^{-f}d\mu + \int_M |\Phi|^{a+1}|\nabla\eta|^2 e^{-f}d\mu \\
&\quad + (a+1)\int_M |\Phi|^a\left\langle\nabla\eta,\nabla|\Phi|\right\rangle\eta e^{-f}d\mu \\
&\quad + \frac{(a+1)^2}{4}\int_M |\Phi|^{a-1}|\nabla|\nabla\Phi||^2\eta^2 e^{-f}d\mu.
\end{aligned}$$

Hence,

$$\begin{aligned}
&\left[a+\frac{1}{n}-\frac{(a+1)^2}{4}\right]\int_M \eta^2|\Phi|^{a-1}|\nabla|\Phi||^2 e^{-f}d\mu \\
&\quad\leq \int_M |\Phi|^{a+1}|\nabla\eta|^2 e^{-f}d\mu + (a+3)\int_M |\Phi|^a\left\langle\nabla\eta,\nabla|\Phi|\right\rangle\eta e^{-f}d\mu. \qquad (5.25)
\end{aligned}$$

From the Cauchy-Schwarz inequality and the inequality $xy \leq \varepsilon x^2 + \frac{1}{4\varepsilon}y^2$ for all $\varepsilon > 0$, we see that

$$\begin{aligned}
(a+3)|\Phi|^a\left\langle\nabla\eta,\nabla|\Phi|\right\rangle\eta &\leq |a+3|\left(|\Phi|^{\frac{a-1}{2}}|\nabla|\Phi||\,|\eta|\right)\left(|\Phi|^{\frac{a+1}{2}}|\nabla\eta|\right) \\
&\leq \varepsilon|\Phi|^{a-1}|\nabla|\Phi||^2\eta^2 + \frac{(a+3)^2}{4\varepsilon}|\Phi|^{a+1}|\nabla\eta|^2. \qquad (5.26)
\end{aligned}$$

Substituting (5.26) into (5.25), we get

$$\begin{aligned}
&\left[a+\frac{1}{n}-\frac{(a+1)^2}{4}-\varepsilon\right]\int_M |\Phi|^{a-1}|\nabla|\Phi||^2\eta^2 e^{-f}d\mu \\
&\qquad\leq \left[1+\frac{(a+3)^2}{4\varepsilon}\right]\int_M |\Phi|^{a+1}|\nabla\eta|^2 e^{-f}d\mu. \qquad (5.27)
\end{aligned}$$

Now let $p = a + 1$. Then, the above inequality becomes

$$\left[p - 1 - \frac{p^2}{4} + \frac{1}{n} - \varepsilon\right] \int_M |\Phi|^{p-2} |\nabla |\Phi||^2 \eta^2 e^{-f} d\mu \leq \left[1 + \frac{(p+2)^2}{4\varepsilon}\right] \int_M |\Phi|^p |\nabla \eta|^2 e^{-f} d\mu. \quad (5.28)$$

Next, we choose the number p to be $p - 1 - \frac{p^2}{4} + \frac{1}{n} > 0$, or equivalently

$$2 - \frac{2}{\sqrt{n}} < p < 2 + \frac{2}{\sqrt{n}} = 2\left(1 + \sqrt{\frac{1}{n}}\right).$$

Hence, for $2 - \frac{2}{\sqrt{n}} < p < 2 + \frac{2}{\sqrt{n}}$, we can choose $\varepsilon > 0$ small such that there is a constant $C > 0$ depending on n, p such that

$$\int_M |\Phi|^{p-2} |\nabla |\Phi||^2 \eta^2 e^{-f} d\mu \leq C \int_M |\Phi|^p |\nabla \eta|^2 e^{-f} d\mu.$$

Let $o \in M$ be a fixed point and let $B_R(o)$ be the geodesic ball centered at o with radius R. We choose η to be a smooth function on M such that $0 \leq \eta \leq 1$. Moreover, η satisfies:

(i) $\eta = 1$ on $B_{\frac{R}{2}}(o)$ and $\eta = 0$ outside $B_{2R}(o)$;

(ii) $|\nabla \eta| \leq \frac{2}{R}$.

Plugging η into the above inequality then letting R tend to infinity, we conclude that $|\nabla |\Phi|| = 0$, since $|\Phi| \in L^p(M_f)$. Therefore, $|\Phi|$ is constant. Note that a translating solition is of Euclidean volume growth ([106]), this implies $\Phi = 0$ because $|\Phi| \in L^p(M_f)$. Now, we apply the argument as in the proof of Theorem 5.1 to conclude that M is a hyperplane. □

As a consequence of this theorem, for $p = 2$, we obtain the following corollary, which can be considered as an improvement of Theorem 6 by Ma and Miquel in [68].

Corollary 5.1. *Let $X : M^{n \geq 2} \to \mathbb{R}^{n+1}$ be a translator with mean curvature which does not change sign and*

$$|\nabla A| \leq \frac{n+1}{2n} |\nabla H|.$$

Suppose that the traceless second fundamental form of the immersion satisfies $|\Phi| \in L^2(M_f)$. Then M is a hyperplane.

5.2 Vanishing result for weighted harmonic forms

In this section, we give a proof of Theorem 5.3.

Proof the Theorem 5.3. Let ω be L^p_f harmonic 1-form on M. We denote the dual vector field of ω by $\omega^\sharp$ and $\|A\|_{n,f} = \left(\int_M |A|^n e^{-f} d\mu\right)^{\frac{1}{n}}$. Applying the extended Bochner formula for a L^p_f harmonic 1-form, we get

$$\begin{aligned}\Delta_f |\omega|^2 &= 2|\nabla\omega|^2 + 2\left\langle \Delta_f \omega, \omega\right\rangle + 2\,\mathrm{Ric}_f\left(\omega^\sharp, \omega^\sharp\right) \\ &= 2|\nabla\omega|^2 + 2\,\mathrm{Ric}_f\left(\omega^\sharp, \omega^\sharp\right). \end{aligned} \tag{5.29}$$

Note that $\Delta_f |\omega|^2 = 2\,|\omega|\,\Delta_f\,|\omega| + 2|\nabla\,|\omega||^2$ and the Bakry-Emery Ricci tensor of M satisfies

$$\mathrm{Ric}_f\left(\omega^\sharp, \omega^\sharp\right) = -\left\langle A^2\omega^\sharp, \omega^\sharp\right\rangle.$$

This implies

$$|\omega|\,\Delta_f\,|\omega| = |\nabla\omega|^2 - |\nabla\,|\omega||^2 - \left\langle A^2\omega^\sharp, \omega^\sharp\right\rangle.$$

Consequently, by Kato's inequality, we have

$$|\omega|\,\Delta_f\,|\omega| \geq -\left\langle A^2\omega^\sharp, \omega^\sharp\right\rangle \geq -\left|A^2\omega^\sharp\right|\left|\omega^\sharp\right| \geq -\,|A|^2\,|\omega|^2.$$

Now, let η be a smooth compactly supported function on M. By multiplying both sides of the above inequality by $\eta^2|\omega|^{p-2}$ and then integrating the obtained result, we arrive at

$$\int_M \eta^2|\omega|^{p-1}\Delta_f\,|\omega|\,e^{-f}d\mu \geq -\int_M |A|^2\,|\omega|^p\eta^2 e^{-f}d\mu. \tag{5.30}$$

Since η has compact support on M, by the Stokes theorem, we see that

$$\begin{aligned}&\int_M \eta^2|\omega|^{p-1}\Delta_f\,|\omega|\,e^{-f}d\mu \\ &= -\int_M \left\langle \nabla\left(\eta^2|\omega|^{p-1}\right), \nabla\,|\omega|\right\rangle e^{-f}d\mu \\ &= -2\int_M |\omega|^{p-1}\left\langle \nabla\eta, \nabla\,|\omega|\right\rangle \eta e^{-f}d\mu - (p-1)\int_M \eta^2|\omega|^{p-2}|\nabla\,|\omega||^2 e^{-f}d\mu.\end{aligned}$$

This inequality and (5.30) implies

$$\begin{aligned}&(p-1)\int_M \eta^2|\omega|^{p-2}|\nabla\,|\omega||^2 e^{-f}d\mu \\ &\quad\leq -2\int_M |\omega|^{p-1}\left\langle \nabla\eta, \nabla\,|\omega|\right\rangle \eta e^{-f}d\mu + \int_M |A|^2|\omega|^p\eta^2 e^{-f}d\mu.\end{aligned} \tag{5.31}$$

By Hölder's inequality and the weighted Sobolev inequality, we have

$$\begin{aligned}
&\int_M |A|^2|\omega|^p\eta^2 e^{-f} d\mu \\
&\leq \left(\int_M |A|^n\right)^{\frac{2}{n}} \left(\int_M \left(\eta|\omega|^{\frac{p}{2}}\right)^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}} \\
&\leq \left(\frac{2C(n)}{n-1}\right)^2 \|A\|^2_{n,f} \int_M \left|\nabla\left(\eta|\omega|^{\frac{p}{2}}\right)\right|^2 e^{-f} d\mu \\
&= D_n\|A\|^2_{n,f} \int_M \Big(|\omega|^p|\nabla\eta|^2 + p|\omega|^{p-1}\langle\nabla|\omega|, \nabla\eta\rangle\eta \\
&\qquad\qquad + \frac{p^2}{4}|\omega|^{p-2}\eta^2|\nabla|\omega||^2\Big) e^{-f} d\mu \qquad (5.32)
\end{aligned}$$

where $D_n = \left(\frac{2C(n)}{n-1}\right)^2$. Using the Cauchy-Schwarz inequality and the inequality $xy \leq \varepsilon x^2 + \frac{y^2}{4\varepsilon}$ for any $\varepsilon > 0$, we see that

$$\begin{aligned}
p|\omega|^{p-1}\langle\nabla|\omega|, \nabla\eta\rangle &\leq p|\omega|^{p-1}|\langle\nabla|\omega|, \nabla\eta\rangle||\eta| \\
&\leq p|\omega|^{p-1}|\nabla|\omega|||\nabla\eta||\eta| \\
&= \frac{|\nabla\eta|^2|\omega|^p}{\varepsilon} + \frac{\varepsilon p^2}{4}|\omega|^{p-2}\eta^2|\nabla|\omega||^2.
\end{aligned}$$

This together with (5.32) implies

$$\begin{aligned}
\int_M |A|^2|\omega|^p\eta^2 e^{-f} d\mu &\leq D_n\|A\|^2_{n,f}\Bigg[\left(1+\frac{1}{\varepsilon}\right)\int_M |\omega|^p|\nabla\eta|^2 e^{-f} d\mu \\
&\qquad + \frac{(1+\varepsilon)p^2}{4}\int_M |\omega|^{p-2}\eta^2|\nabla|\omega||^2 e^{-f} d\mu\Bigg] \\
&= \left(1+\frac{1}{\varepsilon}\right) D_n\|A\|^2_{n,f}\int_M |\omega|^p|\nabla\eta|^2 e^{-f} d\mu \\
&\qquad + \frac{(1+\varepsilon)p^2}{4} D_n\|A\|^2_{n,f}\int_M |\omega|^{p-2}\eta^2|\nabla|\omega||^2 e^{-f} d\mu \qquad (5.33)
\end{aligned}$$

On the other hand, for any $\varepsilon > 0$, we have

$$\begin{aligned}
&-2\int_M |\omega|^{p-1}\langle\nabla\eta, \nabla|\omega|\rangle\eta e^{-f} d\mu \\
&\leq 2\int_M |\omega|^{p-1}|\langle\nabla\eta, \nabla|\omega|\rangle||\eta| e^{-f} d\mu \\
&\leq \frac{1}{\varepsilon}\int_M |\nabla\eta|^2|\omega|^p e^{-f} d\mu + \varepsilon\int_M |\omega|^{p-2}|\nabla|\omega||^2\eta^2 e^{-f} d\mu. \qquad (5.34)
\end{aligned}$$

Combining (5.31), (5.33), and (5.34), we get

$$\left[p-1-\frac{p^2}{4}D_n\left\|A\right\|_{n,f}^2-\frac{\varepsilon p^2}{4}D_n\left\|A\right\|_{n+1,f}^2+\varepsilon\right]\int_M \eta^2|\omega|^{p-2}\left|\nabla\left|\omega\right|\right|^2e^{-f}d\mu$$
$$\leq\left[\left(1+\frac{1}{\varepsilon}\right)D_n\left\|A\right\|_{n,f}^2+\frac{1}{\varepsilon}\right]\int_M|\omega|^p\left|\nabla\eta\right|^2e^{-f}d\mu.$$

For a sufficiently small $\varepsilon>0$, the above inequality implies that there is a constant $C>0$ such that

$$\int_M|\omega|^{p-2}\left|\nabla\left|\omega\right|\right|^2e^{-f}\eta^2d\mu\ \leq C\int_M|\omega|^p\left|\nabla\eta\right|^2e^{-f}d\mu, \tag{5.35}$$

provided that $p-1-\frac{p^2}{4}D_n\left\|A\right\|_{n,f}^2>0$, or equivalently

$$\left\|A\right\|_{n,f}^2<\frac{4\left(p-1\right)}{p^2D_n}=\frac{\left(p-1\right)\left(n-1\right)^2}{p^2C^2\left(n\right)}.$$

Let $o\in M$ be a fixed point and let $B_R(o)$ be the geodesic ball centered at o with radius R. We choose η to be a smooth function on M such that $0\leq\eta\leq1$. Moreover, η satisfies:

(i) $\eta=1$ on $B_{\frac{R}{2}}(o)$ and $\eta=0$ outside $B_R(o)$;

(ii) $|\nabla\eta|\leq\frac{2}{R}$.

Applying this test function η to (5.35), we get

$$\int_{B_R(o)}|\omega|^{p-2}\left|\nabla\left|\omega\right|\right|^2e^{-f}d\mu\ \leq\frac{4C}{R^2}\int_{B_R(o)}|\omega|^pe^{-f}d\mu. \tag{5.36}$$

Letting R tend to ∞ in the above inequality and noting that $\omega\in L_p^f$, we conclude that $\nabla\left|\omega\right|=0$, which shows that $|\omega|$ is a constant. Moreover, since $\int_M|\omega|^pe^{-f}d\mu<\infty$ and the weighted volume of M is infinite, we finally get $\omega=0$. The proof is complete. □

Now, we note that if a Sobolev inequality holds true on M every end of M is non-f-parabolic, for example see [48]. Therefore, we have the following corollary.

Corollary 5.2. *Let $X:M^{n\geq3}\to\mathbb{R}^{n+1}$ be a smooth complete translating soliton in the Euclidean space $\mathbb{R}^{n+1}$ contained in the halfspace*

$$\Pi_{V,a}=\left\{y\in\mathbb{R}^{n+1}:\langle y,V\rangle\geq a\right\},$$

for some $a \in \mathbb{R}$. Furthermore, assume that

$$\left(\int_M |A|^n e^{-f} d\mu\right)^{\frac{1}{n}} \le \frac{n-1}{2S(n)},$$

where $S(n)$ is the constant as in Lemma 5.4. Then there are no nontrivial L^2_f harmonic 1-forms on M. In particular, M has only one end.

Proof of Corollary 5.2. Since every end of M is non-f-parabolic, we can argue by contradiction to assume that M has at least two ends. Then by Li-Tam [62], there exists a non-constant f-harmonic function u such that $\omega := du$ satisfying $|\omega| \in L^2_f$. An application of Theorem 5.3 implies that $\omega = 0$ or u is constant. This is a contradiction. The proof is complete. □

5.3 Translators with a Sobolev inequality

Suppose that M satisfies the following Sobolev inequality

$$\left[\int_M u^{\frac{2(n+1)}{n-1}} \varrho d\mu\right]^{\frac{n-1}{n+1}} \le \left(\frac{2C(n)n}{n-1}\right)^2 \int_M |\nabla u|^2 \varrho d\mu \tag{5.37}$$

for any u that is a non-negative compactly supported C^1 function on M and $C(n)$ is the Sobolev constant. In fact, the above inequality was proved in [48]. However, the authors pointed out in [49] that there is a gap in their proof of this inequality. Here, we assume that this inequality holds true. The Sobolev inequality (5.37) was used by Kunikawa and Saito in [55] to study the injectivity of the natural map between the first de Rham cohomology group with compact support, the reduced L^2_f cohomology, and the space of L^2_f f-harmonic 1-forms. They proved that if M supports the Sobolev inequality (5.37) and admits a codimension one cycle which does not disconnect M then the space of L^2_f f-harmonic 1-forms is non-trivial.

Now, we apply the above Sobolev inequality above to $u = |\Phi|^a\eta$. Then we have

$$\begin{aligned}\left[\int_M \left(|\Phi|^a\eta\right)^{\frac{2(n+1)}{n-1}} \varrho d\mu\right]^{\frac{n-1}{n+1}} &\le \left(\frac{2C(n)n}{n-1}\right)^2 \int_M \left|\nabla\left(|\Phi|^a\eta\right)\right|^2 \varrho d\mu \\ &= \left(\frac{2C(n)n}{n-1}\right)^2 \left(\int_M a^2|\nabla|\Phi||^2|\Phi|^{2a-2}\eta^2\varrho d\mu \right.\\ &\quad \left. + \int_M 2a|\Phi|^{2a-1}\eta\langle\nabla|\Phi|, \nabla\eta\rangle\varrho d\mu + \int_M |\Phi|^{2a}|\nabla\eta|^2\varrho d\mu\right).\end{aligned} \tag{5.38}$$

By the Cauchy inequality, we obtain

$$\begin{aligned}\left[\int_M (|\Phi|^a\eta)^{\frac{2(n+1)}{n-1}}\varrho d\mu\right]^{\frac{n-1}{n+1}} &\leq \left(\frac{2C(n)n}{n-1}\right)^2 \left(\int_M a^2|\nabla|\Phi||^2|\Phi|^{2a-2}\eta^2\varrho d\mu\right.\\ &\quad \left.+\int_M 2a|\Phi|^{2a-1}\eta\langle\nabla|\Phi|,\nabla\eta\rangle\varrho d\mu + \int_M |\Phi|^{2a}|\nabla\eta|^2\varrho d\mu\right)\\ &\leq \left(\frac{2C(n)n}{n-1}\right)^2 \left((1+\delta)\int_M a^2|\nabla|\Phi||^2|\Phi|^{2a-2}\eta^2\varrho d\mu\right.\\ &\quad \left.+\left(1+\frac{1}{\delta}\right)\int_M |\Phi|^{2a}|\nabla\eta|^2\varrho d\mu\right).\end{aligned} \tag{5.39}$$

Apply (5.9) and keep in mind that right now $\iota = 2$. For $0 < \varepsilon < a - \frac{1}{2}$, we have

$$\begin{aligned}\kappa_1^{-1}\left[\int_M (|\Phi|^a\eta)^{\frac{2(n+1)}{n-1}}\varrho d\mu\right]^{\frac{n-1}{n+1}} &\leq \left\{\frac{a^2(1+\delta)}{4\left(a-\frac{1}{2}-\varepsilon\right)}\left(2\int_M |\Phi|^{2a+2}\eta^2\varrho d\mu\right.\right.\\ &\quad \left.+\frac{2}{n}\int_M |\Phi|^{2a}|H|^2\eta^2\varrho d\mu + \frac{1}{\varepsilon}\int_M |\Phi|^{2a}|\nabla\eta|^2\varrho d\mu\right)\\ &\quad \left.+\left(1+\frac{1}{\delta}\right)\int_M |\Phi|^{2a}|\nabla\eta|^2\varrho d\mu\right\},\end{aligned} \tag{5.40}$$

where $\kappa_1 = \left(\frac{2C(n)n}{n-1}\right)^2$.

Using the fact that $|A|^2 = |\Phi|^2 + \frac{1}{n}|H|^2$, we can rewrite (5.40) as

$$\begin{aligned}\kappa_1^{-1}\left[\int_M (|\Phi|^a\eta)^{\frac{2(n+1)}{n-1}}\varrho d\mu\right]^{\frac{n-1}{n+1}} &\leq \frac{a^2(1+\delta)}{2\left(a-\frac{1}{2}-\varepsilon\right)}\int_M |\Phi|^{2a}|A|^2\eta^2\varrho d\mu\\ &\quad + \widetilde{C}(n,a,\delta,\varepsilon)\int_M |\Phi|^{2a}|\nabla\eta|^2\varrho d\mu,\end{aligned} \tag{5.41}$$

where $\widetilde{C}(n,a,\delta,\varepsilon)$ is an explicit positive constant depending on $n, a, \delta, \varepsilon$.

By the Hölder's inequality, we have that

$$\begin{aligned}\int_M |\Phi|^{2a}|A|^2\eta^2\varrho d\mu &\leq \left(\int_M |A|^{2\cdot\frac{n+1}{2}}\varrho d\mu\right)^{\frac{2}{n+1}} \cdot \left(\int_M \left(|\Phi|^{2a}\eta^2\right)^{\frac{n+1}{n-1}}\varrho d\mu\right)^{\frac{n-1}{n+1}}\\ &= \left(\int_M |A|^{n+1}\varrho d\mu\right)^{\frac{2}{n+1}} \cdot \left(\int_M (|\Phi|^a\eta)^{\frac{2(n+1)}{n-1}}\varrho d\mu\right)^{\frac{n-1}{n+1}}.\end{aligned} \tag{5.42}$$

Our goal is to decrease the number of conditions in theorem 5.1, only one condition instead of two as in Theorem 5.1, so we should choose $a = \frac{n+1}{2}$. For that reason, combining (5.41) and (5.42), we have

$$\begin{aligned}
&\kappa_1^{-1}\left[\int_M \left(|\Phi|^{\frac{n+1}{2}}\eta\right)^{\frac{2(n+1)}{n-1}}\varrho d\mu\right]^{\frac{n-1}{n+1}} \\
\leq &\frac{(n+1)^2(1+\delta)}{8\left(\frac{n+1}{2}-\frac{1}{2}-\varepsilon\right)}\left(\int_M |A|^{n+1}\right)^{\frac{2}{n+1}}\cdot\left(\int_M \left(|\Phi|^{\frac{n+1}{2}}\eta\varrho d\mu\right)^{\frac{2(n+1)}{n-1}}\varrho d\mu\right)^{\frac{n-1}{n+1}} \\
&+\widetilde{C}(n,\delta,\varepsilon)\int_M |\Phi|^{n+1}|\nabla\eta|^2\varrho d\mu.
\end{aligned}$$

Put

$$K_1(n,\varepsilon,\delta)=\sqrt{\frac{8\left(\frac{n+1}{2}-\frac{1}{2}-\varepsilon\right)}{(n+1)^2(1+\delta)\kappa_1}},$$

and

$$K_1(n)=\sup_{\delta>0,\,0<\varepsilon<a-\frac{1}{2}}K_1(n,\varepsilon,\delta)=\sqrt{\frac{(n-1)^2}{C(n)^2(n+1)^2n^2}}.$$

Applying the argument as in the proof of Theorem 5.1, we have the following result.

Theorem 5.6. *Let $X: M^{n\geq 3}\to\mathbb{R}^{n+1}$ be a smooth complete translating soliton in the Euclidean space $\mathbb{R}^{n+1}$ with Sobolev inequality (5.37). If the second fundamental form A of M satisfies*

$$\left(\int_M |A|^{n+1}\varrho d\mu\right)^{\frac{1}{n+1}}<K_1(n),$$

where $K_1(n)$ is defined as above, then M is a hyperplane.

Conclusions

The main results of the dissertation include:

1) An upper bound on the dimension of the Lie algebra of Killing vector fields on an irreducible, non-trivial gradient Ricci soliton, as well as some results on the geometric structure of this class of gradient Ricci solitons when this maximal dimension is attained;

2) Liouville-type theorems and gradient estimates for the positive bounded solutions to the nonlinear parabolic equation related to gradient Ricci solitons concerning Perelman's reduced distance along ancient k-super Ricci flow;

3) Some analytical aspects of a general type of nonlinear parabolic equation concerning the weighted Laplacian on a smooth metric measure space, with the metric evolving under the (k, ∞)-super Perelman-Ricci flow and the Yamabe flow, such as gradient estimates, Harnack inequalities, general global constancy, and Liouville type theorems;

4) Rigidity and vanishing results for complete translating solitons in Euclidean spaces.

In the near future, we will focus on researching two key problems that will continue the work done in this dissertation.

1) The main approach to studying Problem 1.1 in this dissertation is to use the level set of the potential function of gradient Ricci solitons. However, in the case of Einstein manifolds (that is, $\mathrm{Ric} = \lambda g$), this approach is not feasible due to the absence of the potential function. We aim to estimate the upper bound on the dimension of the group of isometries of an Einstein manifold and classify the spaces where this maximum dimension is attained. We also intend to study the group of isometries of quasi-Einstein m-manifolds.

   Besides, we are also particularly interested in classifying Kähler gradient Ricci solitons with geometric transformation groups in real dimension four.

2) Let (M^n, g) be an n-dimensional complete Riemannian manifold. For any smooth vector field V on M, the m-Bakry-Émery Ricci tensor is defined by

$$\operatorname{Ric}_V^m := \operatorname{Ric} + \frac{1}{2}\mathcal{L}_V g - \frac{1}{m} V^* \otimes V^*$$

for some number $m > 0$. Here $\mathcal{L}_V$ denotes the Lie derivative in the direction of V, and V^* is the metric dual of V. When $m = 0$, we regard $V \equiv 0$ and Ric_V^m becomes the usual Ricci tensor Ric. When $m = \infty$, we have (∞)-Bakry-Émery Ricci curvature

$$\operatorname{Ric}_V := \operatorname{Ric}_V^\infty = \operatorname{Ric} + \frac{1}{2}\mathcal{L}_V g.$$

We aim to formulate and prove gradient estimates and Hessian estimates for positive smooth solutions u to the following non-linear parabolic equation

$$\left(\frac{\partial}{\partial t} - \Delta_V\right) F(u(x,t)) = G(u(x,t)).$$

Here, Δ_V is the so-called V-Laplacian, which acts on functions $u \in \mathcal{C}^2(M)$ by $\Delta_V u = \Delta u - \langle V, \nabla u\rangle$. From these estimates, we will derive various analytical aspects, such as Harnack inequalities, results on parabolic frequency, and Liouville and global constancy-type results. It should be emphasized that, under the assumption regarding Ric_V, most of the previous gradient estimates require that the smooth vector field V be bounded, that is, $|V| \le a$ for some real constant $a \ge 0$. We hope that this condition can be eliminated in the estimation results.

List of Author's Related Papers

1. Ha Tuan Dung, Hung Tran (2025), "On isometry groups of gradient Ricci solitons", to appear in *Forum Mathematicum* (SCI-E, Q1), https://doi.org/10.1515/forum-2024-0325.

2. Ha Tuan Dung, Nguyen Tien Manh, and Nguyen Dang Tuyen (2023), "Liouville type theorems and gradient estimates for nonlinear heat equations along ancient K-super Ricci flow via reduced geometry", *Journal of Mathematical Analysis and Applications*, Vol. 519 (2), 126836 (SCI-E, Q1).

3. Ha Tuan Dung (2023), "Gradient estimates for a general type of nonlinear parabolic equations under geometric conditions and related problems", *Nonlinear Analysis*, Vol. 226, 113135 (SCI-E, Q1).

4. Ha Tuan Dung, Nguyen Thac Dung, and Tran Quang Huy (2023), "Rigidity and vanishing theorems for complete translating solitons", *Manuscripta Mathematica*, Vol. 172, pp. 331-352 (SCI-E, Q2).

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