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SOME ASPECTS OF GEOMETRIC FLOWS IN RIEMANNIAN GEOMETRY

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SUMMARY

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Chapter 1

Introduction

The field of geometric flows is one of the most important areas of geometric analysis, forming at the nexus of differential equations and geometry. This field of study is characterized by the deformation of geometric objects such as metrics, mappings, and submanifolds by geometric attributes such as curvature and consists of partial differential equations (PDEs) of parabolic type. These flows have wide applications in many scientific fields. For example, in cell biology, they aid in understanding dynamic network rewiring during cellular differentiation and cancer; in medical imaging, they used to conformal brain mapping and virtual colonoscopy; in computer graphics, they help model vorticity lines for efficient smoke and dust animations in games and CGI effects; and in physics, they can model dynamic systems and space-time geometries. In pure mathematics, geometric flows have demonstrated their great potential by solving various problems related to differential geometry and topology. The field of geometric flows can be seen as a bridge between analysis and geometry. Moreover, thanks to this intersection, researchers can use tools and methods from the theory of PDEs, differential geometry, or both to study challenging problems in this field.

In the PDEs theory, investigating special solutions, such as radial or stable solutions, plays an important role in establishing qualitative and quantitative properties for the general solutions of the equation under consideration. These solutions are either expressible in closed form or, if not feasible, will be systematically classified. Solitons in geometric flows are a typical example of such special solutions. They remain invariant in time to a certain degree under a particular flow. A basic example of these solitons would be a family of round spheres in Euclidean space, which gradually shrink in size over time and eventually collapse to a single point. This behavior serves as a solution to the mean curvature flow, a type of geometric flow that evolves shapes by smoothing them out. On the other hand, as the geometric flow progresses, it can lead to intricate geometric changes, including the appearance of singularities, where quantities containing the norm of the curvature tensor approach to infinity, typically forming in finite time, due in part to the nonlinearity of geometric flow equations, as well as for geometric and topological reasons. Solitons of some geometric flows, such as Ricci flows and mean curvature flows, serve as prototypical singularity models. This is also one of the main motivations to promote further research by mathematicians in this topic and the field of geometric flows in general.

This dissertation investigates some aspects of geometric flows, with a particular focus on two main research directions as follows.

- The first aim is to study some geometric and topological properties of gradient Ricci solitons and translating solitons.
- The second aim is to explore the analytical aspects of some partial differential equations that originate from geometry within the context of some super geometric flows.

In the following three subsections of this chapter, we will provide an overview of the problems studied in the dissertation.

1.1 Gradient Ricci soltions and isometry groups

The Ricci flow equation is a geometric evolution equation that deforms the metric g of a Riemannian manifold over time by adjusting it in a way proportional to the Ricci curvature Ric:

$$\frac{\partial g}{\partial t} = -2\mathrm{Ric}. \tag{1.1}$$

A Ricci flow (or a solution to the above equation) is a one-parameter family of metrics g, defined on a smooth manifold M and parameterized by t within a non-degenerate interval I, that satisfies the equation (1.1). The Ricci flow was introduced in 1982 by Hamilton as part of his ambitious program to prove Poincaré's conjecture and Thurston's geometrization conjecture. It's important to recognize that the Ricci flow equation is only weakly parabolic, which frequently leads to finite-time singularities. This has prompted the study of singularity models to gain insight into the underlying topological and geometric features of Ricci flows. Probably the most important singularity model is the Ricci soliton, which is a self-similar solution to the Ricci flow equation (1.1) and arises as a finite-time singularity model. Recall that a Ricci soliton is a Riemannian manifold (M, g) that is equipped with a smooth vector field X satisfying the equation

$$\mathrm{Ric} + \frac{1}{2}\mathcal{L}_X g = \lambda g, \tag{1.2}$$

where $\mathcal{L}$ is the Lie derivative with respect to X and $\lambda \in \mathbb{R}$. In particular, if $X = \nabla f$ where $f : M \to \mathbb{R}$ is a smooth function, then we say that a triple (M, g, f) is a gradient Ricci soliton. In this case the equation (1.2) becomes

$$\mathrm{Ric} + \mathrm{Hess}\, f = \lambda g, \tag{1.3}$$

where Hess is the Hessian of metric g. Depending on the value of λ, a gradient Ricci soliton is called shrinking if $\lambda > 0$, steady if $\lambda = 0$, or expanding if $\lambda < 0$.

We define a gradient Ricci soliton to be rigid if it is a flat bundle $N \times_\Gamma \mathbb{R}^k$ where N is Einstein, Γ acts freely on N and by orthogonal transformations on $\mathbb{R}^k$ (no translational components) to get a flat vector bundle over a base that is Einstein and with $f = \frac{\lambda}{2}d^2$. Here, d is the distance in the flat fibers to the base. While (non-gradient) Ricci solitons have been found in various Lie groups and homogeneous spaces, Petersen and Wylie proved that all homogeneous gradient Ricci solitons are rigid. Furthermore, they also demonstrated that if the Riemannian metric is reducible, then the soliton structure is also reducible. Their result is based on the existence of splitting results induced by Killing vector fields.

Inspired by Petersen and Wylie's work, in Chapter 2, we will study the isometry group $\mathrm{Iso}(M)$ and its Lie algebra of an irreducible non-trivial gradient Ricci soliton (M, g, f). Recall that a Riemannian manifold is said to be irreducible if no finite cover of it can be expressed (in the isometric sense) as a direct product of manifolds of smaller dimensions.

Problem 1. *Find an upper bound on the dimension of the Lie algebra of Killing vector fields on an irreducible non-trivial gradient Ricci soliton, and classify the spaces where this maximal dimension is attained.*

1.2 Nonlinear parabolic equations and super geometric flows

Turning the framework of geometric flow theory, we now present the concept of super Ricci flow, which was originally introduced by McCann and Topping from the perspective of optimal transport theory. A smooth manifold $(M, g(x,t))_{t \in I}$ is called a super Ricci flow if

$$\frac{\partial g}{\partial t} \geq -2\text{Ric}. \tag{1.4}$$

For each $k \in \mathbb{R}$, a time-dependent Riemannian manifold $(M, g(x,t))_{t\in I}$ is termed a k-super Ricci flow if it satisfies the following condition:

$$\frac{\partial g}{\partial t} + 2\,\text{Ric} \geq 2kg, \tag{1.5}$$

which is a natural extension of the concept of super Ricci flow. A k-supper Ricci flow $(M, g(x,t))_{t\in I}$ is said to be ancient when $I = (-\infty, 0]$.

The reduced distance and reduced volume were first introduced by Perelman in his groundbreaking paper as two key tools for analyzing the Ricci flow. Later, Ye proved several properties of Perelman's reduced distance and obtained some estimates for the reduced volume. Besides, the applications of these properties in the analysis of the asymptotic limits of κ-solutions of the Ricci flow have been presented by Ye in the follow-up paper. Recently, Kunikawa and Sakurai obtained Liouville type theorems for harmonic maps under ancient super Ricci flow with controlled growth, approaching the topic from Perelman's reduced geometric perspective.

The next chapter of this thesis is also motivated from a work due to Ma. For some constants a, b, Ma considered the following nonlinear elliptic equation

$$\Delta u + au \ln u + bu = 0 \tag{1.6}$$

in a complete noncompact Riemannian manifold. From Ma's observation, we know that the above equation is closely related to the equation (1.3) of the gradient Ricci soliton (M, g, f). Moreover, the equation (1.6) is naturally linked to geometric and functional inequalities on manifolds, particularly the logarithmic Sobolev inequality and Perelman's $\mathcal{W}$-entropy. Replacing u by $e^{\frac{b}{a}}u$, we see that the equation (1.6) is equivalent to the following equation

$$\Delta u + au \ln u = 0. \tag{1.7}$$

Inspired by the works of Kunikawa, Sakurai, and Ma, in Chapter 3, we will study gradient estimates for positive bounded solutions to the parabolic counterpart of equation (1.7) along ancient k-super Ricci flow and explore some of its applications. Specifically, we are interested in the following problem.

Problem 2. *Establish gradient estimates and Liouville type results for positive bounded solutions of the nonlinear parabolic equation related to Perelman's reduced distance*

$$\frac{\partial}{\partial t}u(x,t) = \Delta u(x,t) + au(x,t)\ln u(x,t) \tag{1.8}$$

along ancient k-super Ricci flow, where $a \in \mathbb{R}$.

A smooth metric measure space, also known as a weighted manifold or a manifold with density, can be viewed as a natural generalization of gradient Ricci solitons. Since Perelman's works, this space has been the subject of extensive study by many mathematicians worldwide. Recall that a smooth metric measure space is a triple $(M, g, e^{-f}d\mu)$, where (M, g) is a complete Riemannian manifold of dimension $n \geq 3$ endowed with a weighted measure $e^{-f}d\mu$ for some $f \in \mathcal{C}^\infty(M)$ and $d\mu$ is the standard Riemannian volume measure of metric g. On $(M, g, e^{-f}d\mu)$, the weighted Laplacian Δ_f is defined by

$$\Delta_f \cdot := \Delta \cdot - \langle \nabla f, \nabla \cdot \rangle,$$

which is a natural generalization of the Laplace-Beltrami operator Δ to the smooth metric measure space context, and it coincides with the latter precisely when the potential f is a constant function. For any real number $m \geq 0$, the m-Bakry-Émery curvature is defined by

$$\mathrm{Ric}_f^m := \mathrm{Ric} + \mathrm{Hess} f - \frac{1}{m} df \otimes df.$$

When $m = 0$, it means that f is constant and Ric_f^m becomes the usual Ricci curvature Ric. When $m \to \infty$, we have the (∞-)Bakry-Émery Ricci curvature

$$\mathrm{Ric}_f := \mathrm{Ric}_f^\infty = \mathrm{Ric} + \mathrm{Hess} f.$$

It is not difficult to see that $\mathrm{Ric}_f^m \geq c$ infers $\mathrm{Ric}_f \geq c$, but the contrary may not be accurateaccurate. When Ric_f is bounded from below, many geometric properties of manifolds with the Ricci tensor bounded from below were also possibly extended to smooth metric measure spaces, but some extra assumptions on f are required.

Motivated by the above works of Hamilton, McCann-Topping, and Perelman's work for the modified Ricci flow (this flow is often referred to as the Perelman-Ricci flow), X.-D. Li et al. introduced the concept (k, m)-super Perelman-Ricci flow on manifolds equipped with time-dependent metrics and potentials. For $k, m \in \mathbb{R}$ and $m \geq 0$, a time-dependent smooth metric measure space $\left(M, g(x,t), e^{-f(x,t)}d\mu\right)_{t\in I}$ is called (k,m)-super Perelman-Ricci flow if

$$\frac{\partial g}{\partial t} + 2\,\mathrm{Ric}_f^m \geq -2kg. \tag{1.9}$$

When $m \to \infty$, i.e., if the metric $g(x,t)$ and the potential function $f(x,t)$ satisfy the following inequality

$$\frac{\partial g}{\partial t} + 2\mathrm{Ric}_f \geq -2kg, \tag{1.10}$$

we call $\left(M, g(x,t), e^{-f(x,t)}d\mu\right)_{t\in I}$ a (k,∞)-super Perelman-Ricci flow, which can be viewed as a natural extended of the modified Ricci flow.

The Yamabe flow was initially explored by Hamilton in the unpublished work as a means of addressing the Yamabe problem. An n-dimensional manifold $(M, g(x,t))_{t\in I}$ equipped with a time-dependent metric is referred to as a Yamabe flow when it satisfies the following equation

$$\frac{\partial g}{\partial t} = -Sg, \tag{1.11}$$

where S is the scalar curvatures of the metric g. Chow studied the normalized Yamabe flow and demonstrated that this flow converges to a metric with constant scalar curvature. By assuming only that the initial metric is locally conformally flat, Ye established the convergence of the Yamabe flow, thereby improving upon Chow's result. The scenario of metrics that are not conformally flat has been studied in a series of papers by Schwetlick and Struwe, and subsequently by Brendle.

Inspired the work presented in Chapter 3 and the advancements made in the smooth metric spaces discussed earlier, Chapter 4 will concentrate on investigating the following problem. Inspired by the work presented in Chapter 3 and the advancements made in the smooth metric spaces discussed earlier, Chapter 4 will investigate the following problem.

Problem 3. *Study some analytical aspects of a general type of nonlinear parabolic equation concerning the weighted Laplacian*

$$\left(\frac{\partial}{\partial t} - a(x,t) - \Delta_f\right) u(x,t) = F(u(x,t)) \tag{1.12}$$

on a smooth metric measure space with the metric evolving under the (k, ∞)*-super Perelman-Ricci flow* (1.10) *and the Yamabe flow* (1.11)*, where* $a(x,t)$ *is a function which is* $\mathcal{C}^2$ *in the* x*-variable and* $\mathcal{C}^1$ *in the* t*-variable, and* $F(u)$ *is a* $\mathcal{C}^2$ *function of* u*.*

1.3 Translating solitons of the mean curvature flow

We now recall the definition of mean curvature flow. Let $X : M^n \to \mathbb{R}^{n+m}$ be a smooth immersion of an n-dimensional smooth manifold in Euclidean space $\mathbb{R}^{n+m}$. A smooth one-parameter family $X_t = X(\cdot, t)$ of immersions $X_t : M \times [0, T) \to \mathbb{R}^{m+n}$ with corresponding images $M_t = X_t(M)$ is called the mean curvature flow for a submanifold M in $\mathbb{R}^{m+n}$ if it satisfies the following condition

$$\begin{cases} \frac{d}{dt}X(x,t) = H(x,t), \\ \quad X(x,0) = X(x), \end{cases} \tag{1.13}$$

for any $(x,t) \in M \times [0,T)$, where $H(x,t)$ is the mean curvature vector of M_t at $X_t(x)$ in $\mathbb{R}^{m+n}$.

One of the key aspects of studying mean curvature flow is the analysis of singularities. In various scenarios, the second fundamental form with respect to the family M_t may experience singularities. For instance, if M is compact, the second fundamental form will blow up in a finite time. Based on the blow-up rate of the second fundamental form, we categorize the singularities of mean curvature flow into two types: Type-I singularities and Type-II singularities. The geometry of the solution near Type-II singularities is more challenging to control, making the study of Type-II singularities significantly more complex than that of Type-I singularities.

A solution to (1.13) is said to be a translating soliton (or simply a translator) if there exists a constant vector V with unit length in $\mathbb{R}^{n+m}$ such that

$$H = V^{\perp}, \tag{1.14}$$

where $V^{\perp}$ denotes the normal component of V in $\mathbb{R}^{n+m}$. Translating solitons are significant in the theory of mean curvature flow because they arise as blow-up solutions at type II singularities. On the other hand, every translating soliton is a special solution that moves only in a constant direction V without deforming its shape under the mean curvature flow, specifically, the solution is given by $M_t = M + tV$. There are few examples of translating solitons even in the hypersurface case. The primary examples are those translating solitons that are also minimal hypersurfaces. Indeed, by (1.14) we know that V must be tangential to the translator. Consequently, these solitons could have the form of $\widetilde{M} \times L$, where L is a line parallel to V and $\widetilde{M}$ is a minimal hypersurface in $L^{\perp}$.

Inspired by previous works on on translating solitons, in Chapter 5 of this thesis, we are interested in the following problem.

Problem 4. *Study of the rigidity properties and connectedness at infinity of complete translating solitons in the Euclidean space via the second fundamental form.*

1.4 Structure of the present work

As mentioned earlier, the dissertation is divided into five chapters. In addition to Chapter 1, the remaining four chapters will be described below. It also includes a section listing the author's related papers, a Conclusions section, and a list of references. Below is a brief overview of the contents of each chapter, from Chapter 2 to Chapter 5.

In Chapter 2 of this dissertation, we investigate the isometry group $\mathrm{Iso}(M)$ and its Lie algebra of an irreducible non-trivial gradient Ricci soliton (M, g, f). This chapter aims to study Problem 1, which is based on the paper to appear in `https://doi.org/10.1515/forum-2024-0325`.

Chapter 3 of this dissertation is devoted to studying the nonlinear parabolic equation (1.8) related to Perelman's reduced distance, along ancient k-super Ricci flow. This chapter aims to study Problem 2, which is based on the paper published in the *Journal of Mathematical Analysis and Applications*.

In Chapter 4 of this dissertation, we focus instead on studying the general type of nonlinear parabolic equation (1.12) on a smooth metric measure space with the metric evolving under the (k, ∞)-super Perelman-Ricci flow (1.10) and the Yamabe flow (1.11). Chapter 4 aims to study Problem 3, based on the paper published in *Nonlinear Analysis*.

Chapter 5 of this dissertation focuses on studying some aspects of complete translating solitons in the Euclidean space. Chapter 5 aims to study Problem 4, which is based on the paper published in *Manuscripta Mathematica*.

The results of this dissertation were presented at

- The weekly seminar of Geometric Analysis group (June 28, 2023, Vietnam Institute for Advanced Studies in Mathematics, Hanoi);

- The monthly seminar of the Department of Geometry, (December 12, 2023, Hanoi National University of Education, Hanoi);
- The 10th Vietnam Mathematical Congress, Committee on Partial Differential Equations (August 11, 2023, the University of Da Nang-University of Science and Education, Da Nang);
- The Workshop "Some selected topics in Geometric Analysis and applications" (February 1, 2024, Hanoi University of Civil Engineering, Hanoi).

Chapter 2

On isometry groups of gradient Ricci solitons

This chapter is written based on the paper "Ha Tuan Dung, Hung Tran (2025), On isometry groups of gradient Ricci solitons, to appear in *Forum Mathematicum*, `https://doi.org/10.1515/forum-2024-0325`" and focuses on examining Problem 1 discussed in Chapter 1. We specifically investigate the isometry group and its Lie algebra of an irreducible, non-trivial gradient Ricci soliton (M, g, f). Our goal is to determine the maximum dimension of the isometry group and study the structure of this manifold when the maximal dimension is attained. Towards that end, we recall the Lie algebra of the isometry group of (M, g, f):

$$\mathfrak{iso}(M, g) := \{X \text{ is a smooth tangent vector field on } M, \mathcal{L}_X g = 0\}.$$

Closely related to the Lie algebra $\mathfrak{iso}(M, g)$ is the Lie algebra of Killing vector fields preserving f:

$$\mathfrak{iso}_f(M, g, f) := \{X \text{ is a smooth tangent vector field on } M, \mathcal{L}_X g = 0 = \mathcal{L}_X f\}.$$

Throughout this chapter, for convenience in presentation, we will abbreviate the term gradient Ricci soliton as GRS.

In order to achieve the main goal, we first give a result estimating the dimension of $\mathfrak{iso}_f(M, g, f)$ and classify the spaces where this maximal dimension is achieved.

Theorem 2.1. *Let (M^n, g, f), with $n \geq 3$, be a GRS. If f is non-constant then $\mathfrak{iso}_f(M, g, f)$ is of dimension at most $\frac{1}{2}(n-1)n$ and equality happens iff each connected component of a regular level set of f is a space of constant curvature.*

Let $(\mathbb{N}^{n-1}, g_{\mathbb{N}})$ denote the space form model. If $g_{\mathbb{N}}$ is non-flat, the equality happens iff the metric is locally a warped product. That is, there is an open dense *subset such that around each of its points, there is a neighborhood diffeomorphic to a product $I \times \mathbb{N}$ and the metric g is given by $g = dt^2 + F^2(t) g_{\mathbb{N}}$. Here, I is an open interval, and $F : I \mapsto \mathbb{R}^+$ is a smooth function.*

Furthermore, it is possible to relax the assumption on preserving f. A Riemannian manifold is locally irreducible if it is not a local Riemannian product metric around each point.

Theorem 2.2. *Let (M^n, g, f), with $n \geq 3$, be a locally irreducible non-trivial GRS. Then $\mathfrak{iso}(M, g)$ is of dimension at most $\frac{1}{2}(n-1)\,n$. In addition, equality happens iff it is smoothly constructed, as in the case of equality of Theorem 2.1.*

The above theorems are essentially local. That is, there is no mention of the completeness of the metric. Indeed, the soliton structure is so rigid that it is difficult to complete the above metrics.

Theorem 2.3. *Let (M^n, g, f), with $n \geq 3$, be an irreducible non-trivial complete GRS. Then $\mathfrak{iso}(M, g)$ is of dimension at most $\frac{1}{2}(n-1)\,n$. For $\lambda \geq 0$, equality happens iff $\lambda = 0$ and it is isometric to a Bryant soliton.*

Chapter 3

Liouville type theorems and gradient estimates for nonlinear heat equations along ancient k-super Ricci flow via reduced geometry

Recall that for each real number k, a time-dependent Riemannian manifold $(M, g(x,t))_{t\in I}$ is called a k-super Ricci flow if it satisfies the following condition

$$\frac{\partial g}{\partial t} + 2\,\mathrm{Ric} \geq 2kg. \tag{3.1}$$

A k-super Ricci flow $(M, g(x,t))_{t\in I}$ is said to be ancient when $I = (-\infty, 0]$. Written based on the paper "Ha Tuan Dung, Nguyen Tien Manh, and Nguyen Dang Tuyen (2023), Liouville type theorems and gradient estimates for nonlinear heat equations along ancient K-super Ricci flow via reduced geometry, *Journal of Mathematical Analysis and Applications*, Vol. 519 (2), 126836", Chapter 2 delves into the study of Liouville type theorems and gradient estimates for the positive bounded solutions to the nonlinear parabolic equation concerning Perelman's reduced distance

$$\frac{\partial}{\partial t}u(x,t) = \Delta u(x,t) + au(x,t)\ln u(x,t) \tag{3.2}$$

along ancient k-super Ricci flow $(M, g(x,t))_{t\in(-\infty,0]}$, where a is a real number. This is the content of Problem 2 that was discussed in Chapter 1. We will work on the reverse time parameter $\tau := -t$. On this parameter, the ancient K-super

Ricci flow $(M, g(t))_{t\in(-\infty,0]}$ becomes backward k-super Ricci flow $(M, g(\tau))_{\tau\in[0,\infty)}$, namely,

$$\mathrm{Ric} \geq \frac{1}{2}\frac{\partial g}{\partial \tau} + kg.$$

Moreover, the equation (3.2) can be translated as follows

$$\left(\frac{\partial}{\partial \tau} + \Delta\right) u(x,t) = -au(x,t)\ln u(x,t). \tag{3.3}$$

We begin by providing the definition of reduced distance.

Definition 3.1. *The $\mathcal{L}$-length of a curve $\gamma : [\tau_1, \tau_2] \to M$ is defined as*

$$\mathcal{L}(\gamma) := \int_{\tau_1}^{\tau_2} \sqrt{\tau}\left(H + \left|\frac{d\gamma}{d\tau}\right|^2\right) d\tau,$$

where

$$\mathfrak{h} := \frac{1}{2}\partial_\tau g, H := \mathrm{tr}\,\mathfrak{h}.$$

Definition 3.2. *For each $(x,\tau) \in M \times (0,\infty)$, we define the L-distance $L(x,\tau)$ and the reduced distance $\rho(x,\tau)$ from a space-time base point $(x_0, 0)$ as follows*

$$L(x,\tau) := \inf_{\gamma} \mathcal{L}(\gamma), \quad \rho(x,\tau) := \frac{1}{2\sqrt{\tau}}L(x,\tau), \tag{3.4}$$

where we take the infimum over all curves $\gamma : [0,\tau] \to M$ with $\gamma(0) = x_0$ and $\gamma(\tau) = x$. If a curve attains the infimum of (3.4) then it is called minimal $\mathcal{L}$-geodesic from $(x_0, 0)$ to (x, τ).

Definition 3.3. *Let $(M, g(x,\tau))_{\tau\in[0,\infty)}$ be a complete, time-dependent Riemannian manifold. If for each $\tau > 0$ there is $c^\tau \geq 0$ depending only on τ such that $h \geq -c^\tau g$ on $[0,\tau]$ then $(M, g(x,\tau))_{\tau\in[0,\infty)}$ is admissible.*

Note that if $H \geq 0$ then by Definition 3.1, we deduce that $\mathcal{L}$ is non-negative, so is $\rho(x,\tau)$. From this observation, for $(x,\tau) \in M \times (0,\infty)$ and $H \geq 0$, we can define

$$\overline{L}(x,\tau) := 4\tau\rho(x,\tau) = \mathfrak{d}(x,\tau)^2.$$

To establish main results, we will use the following Müller quantity $\mathcal{D}(X)$ and trace Harnack quantity $\mathcal{H}(X)$:

$$\begin{aligned}\mathcal{D}(X) &:= \partial_\tau H - \Delta H - 2|\mathfrak{h}|^2 + 4\,\mathrm{div}\,\mathfrak{h}(X) \\ &\quad - 2g(\nabla H, X) + 2\mathrm{Ric}(X,X) - 2\mathfrak{h}(X,X), \end{aligned} \tag{3.5}$$

$$\mathcal{H}(X) := -\partial_\tau H - \frac{H}{\tau} - 2g(\nabla H, X) + 2\mathfrak{h}(X,X), \tag{3.6}$$

where X is a (time-dependent) vector field.

Our first main result is the following Hamilton type gradient estimate:

Theorem 3.4. *For $k \geq 0$, let $(M, g(x,\tau))_{\tau\in[0,\infty)}$ be an n-dimensional, admissible, complete backward $(-k)$-super Ricci flow. We assume*

$$\mathcal{D}(X) \geq -2k\left(H + |X|^2\right), \quad \mathcal{H}(X) \geq -\frac{H}{\tau}, \quad H \geq 0,$$

for all vector fields X. Let $u : M \times [0,\infty) \to (0,\infty)$ be a positive solution to backward nonlinear heat equation (3.3). *For $R, T > 0$ and $B > 0$, we suppose $u \leq B$ in the cylinder $\mathcal{Q}_{R,T}$. Then there exists a positive constant $c = c(n)$ depending only on n such that*

$$\frac{|\nabla u|}{u} \leq c\left(\frac{\sqrt{A}}{R} + \frac{1}{\sqrt{T}} + \sqrt{k} + \sqrt{\sup_{\mathcal{Q}_{R,T}}\left\{[a(2+2\ln B - \ln u)]^+\right\}}\right)\sqrt{1+\ln\frac{B}{u}} \tag{3.7}$$

in $\mathcal{Q}_{\frac{R}{2},\frac{T}{4}}$, where $A = 1 + \ln B - \ln\left(\inf_{\mathcal{Q}_{R,T}} u\right)$.

When $a = 0$, we can derive the following local space-only gradient estimate for the backward heat equation under the $(-k)$-super Ricci flow.

Corollary 3.5. *For $k \geq 0$, let $(M, g(x,\tau))_{\tau\in[0,\infty)}$ be an n-dimensional, admissible, complete backward $(-k)$-super Ricci flow. We assume*

$$\mathcal{D}(X) \geq -2k\left(H + |X|^2\right), \quad \mathcal{H}(X) \geq -\frac{H}{\tau}, \quad H \geq 0,$$

for all vector fields X. Let $u : M \times [0,\infty) \to (0,\infty)$ stands for a positive solution to the backward heat equation

$$\left(\frac{\partial}{\partial\tau} + \Delta\right) u = 0. \tag{3.8}$$

For $R, T > 0$ and $B > 0$, we suppose $u \leq B$ in the cylinder $\mathcal{Q}_{R,T}$. Then there *exists a positive constant $c = c(n)$ depending only on n such that*

$$\frac{|\nabla u|}{u} \leq c\left(\frac{\sqrt{A}}{R} + \frac{1}{\sqrt{T}} + \sqrt{k}\right)\sqrt{1 + \ln\frac{B}{u}}, \tag{3.9}$$

in $\mathcal{Q}_{\frac{R}{2},\frac{T}{4}}$, where $A = 1 + \ln B - \ln\left(\inf_{\mathcal{Q}_{R,T}} u\right)$.

As an application of Theorem 3.4, we have the following Liouville theorem for the backward nonlinear heat equation (3.3).

Theorem 3.6. *Let $(M, g(x,\tau))_{\tau\in[0,\infty)}$ be an n-dimensional, admissible, complete backward super Ricci flow. We assume*

$$\mathcal{D}(X) \geq 0, \quad \mathcal{H}(X) \geq -\frac{H}{\tau}, \quad H \geq 0 \tag{3.10}$$

for all vector fields X.

1. *When $a < 0$, let $u : M \times [0,\infty) \to (0,\infty)$ be a positive solution to backward nonlinear heat equation (3.3). If $e^{-2} \leq u \leq B$ for some constant $B < 1$, then u does not exist; if $e^{-2} \leq u \leq B$ for some constant $B \geq 1$, then $u \equiv 1$.*
2. *When $a = 0$:*

   2a. *If $u : M \times [0,\infty) \to (0,\infty)$ be a positive solution to backward heat equation (3.8) such that*

   $$u(x,\tau) = \exp\left[o\left(\mathfrak{d}(x,\tau) + \tau\right)\right] \tag{3.11}$$

   near infinity, then u is constant.

   2b. *If $u : M \times [0,\infty) \to \mathbb{R}$ be a solution to backward heat equation (3.8) such that*

   $$u(x,\tau) = o\left(\mathfrak{d}(x,\tau) + \sqrt{\tau}\right) \tag{3.12}$$

   near infinity, then u is constant.

Chapter 4

Gradient estimates for a general type of nonlinear parabolic equations under geometric conditions and related problems

This chapter is written based on the paper "Ha Tuan Dung (2023), Gradient estimates for a general type of nonlinear parabolic equations under geometric conditions and related problems, *Nonlinear Analysis*, Vol. 226, 113135". In the present chapter, we establish gradient estimates for the positive bounded solutions to a general type of nonlinear parabolic equation concerning the weighted Laplacian

$$\left(\frac{\partial}{\partial t} - a(x,t) - \Delta_f\right) u(x,t) = F(u(x,t)) \tag{4.1}$$

on a smooth metric measure space with the metric evolving under the (k,∞)-super Perelman-Ricci flow (1.10) and the Yamabe flow (1.11), where $a(x,t)$ is a function which is $\mathcal{C}^2$ in the x-variable and $\mathcal{C}^1$ in the t-variable, and $F(u)$ is a $\mathcal{C}^2$ function of u. We derive several outcomes from these estimates, including Harnack inequalities, general global constancy, and Liouville type theorems. Applications related to some important geometric partial differential equations are presented to illustrate the strength of the results. The content of this chapter can be seen as a continuation of the work done previously in Chapter 3.

In order to state the main results in Chapter 4, we introduce some notations. On an n-dimensional smooth metric measure space $\left(M, g(x,t), e^{-f(x,t)}d\mu\right)_{t\in[0,T]}$ with the metric evolving under the geometric flow, we write $\operatorname{dist}(x, x_0, t)$ (or $r(x,t)$) for the Riemannian distance between $x \in M$ and x_0 with respect to the metric $g(x,t)$, where $x_0 \in M$ is a fixed point. We introduce the compact set

$$\mathcal{Q}_{R,T} := \{(x,t) \in M \times [0,T] \mid \operatorname{dist}(x, x_0, t) \leq R\},$$

where $R \geq 2$ and $T > 0$. We make use of the following notations $q^+ := \max\{q, 0\}$, $q^- := \min\{q, 0\}$, and

$$\mu = \max_{(x,t)} \{\Delta_f r(x,t) : \operatorname{dist}(x, x_0, t) = 1, 0 \leq t \leq T\}, \quad \mu^+ := \max\{\mu, 0\}.$$

On the static metric measure space $(M, g, e^{-f}d\mu)$, let $d(x, x_0)$ (or $r(x)$) denote the Riemannian distance to x from x_0 with respect to g, and $B(x_0, R)$ denote the geodesic ball centered at x_0 of radius $R \geq 2$. For $T > 0$, let $Q_{R,T}$ be

$$Q_{R,T} := B(x_0, R) \times [0,T] \subset M \times [0,\infty).$$

In this case, we also introduce the following quantities $\mu := \max_{\{x|d(x,x_0)=1\}} \Delta_f r(x)$, $\mu^+ := \max\{\mu, 0\}$.

Our first main result states as follows.

Theorem 4.1. *Let $\left(M, g(x,t), e^{-f(x,t)}d\mu\right)_{t\in[0,T]}$ be a complete solution to the (k,∞)-super Perelman-Ricci flow* (1.10) *on an n-dimensional smooth manifold M and u be a smooth positive solution to the nonlinear heat equation* (4.1) *in $\mathcal{Q}_{R,T}$. Assume that $0 < u \leq B$ and*

$$\operatorname{Ric}_f \geq -(n-1)K, \quad \frac{\partial g}{\partial t} \geq -2Hg$$

for some $K, H \geq 0$ in $\mathcal{Q}_{R,T}$. Then there exists a constant c depending only n such that

$$\frac{|\nabla u|}{u} \leq c\left[\frac{\sqrt{A}}{R} + \sqrt{\frac{\mu^+}{R}} + \frac{1}{\sqrt{t}} + \sqrt[4]{\left(k^+\right)^2 + K^2 + H^2 + \mathcal{P}^2} + \Gamma_a\right]\sqrt{1 + \ln\frac{B}{u}} \tag{4.2}$$

for all $(x,t) \in \mathcal{Q}_{\frac{R}{2},T}$ with $t \neq 0$, where $A = 1 + \ln B - \ln\left(\inf_{\mathcal{Q}_{R,T}} u\right)$ and

$$\Gamma_a = \sup_{\mathcal{Q}_{R,T}} \left\{\left(a^+\right)^{\frac{1}{2}} + |\nabla a|^{\frac{1}{3}}\right\},$$

$$\mathcal{P}=\sup_{\mathcal{Q}_{R,T}}\left\{\left[2F'(u)-\frac{2F(u)}{u}+\frac{1}{1+\ln B-\ln u}\frac{F(u)}{u}\right]^{+}\right\}.$$

In the static case $\frac{\partial g}{\partial t}\equiv 0$ and $\frac{\partial f}{\partial t}\equiv 0$, we can set $H=0$ and $k=(n-1)K$ in Theorem 4.1. Then, we see that $(M,g,e^{-f}d\mu)$ becomes a static smooth metric measure space where $\mathrm{Ric}_f\geq -(n-1)K$ for some constant $K\geq 0$ in the geodesic ball $B(x_0,R)$. From this observation and Theorem 4.1, we have the following result.

Theorem 4.2. *Let $(M,g,e^{-f}d\mu)$ be an n-dimensional complete smooth metric measure space with $\mathrm{Ric}_f\geq -(n-1)K$ for some constant $K\geq 0$ in $B(x_0,R)$. Assume that $0<u(x,t)\leq B$ for some constant B, is a smooth solution to the nonlinear heat equation* (4.1) *in $Q_{R,T}$. Then there exists a constant c depending only n such that*

$$\frac{|\nabla u|}{u}\leq c\left[\frac{\sqrt{A}}{R}+\sqrt{\frac{\mu^{+}}{R}}+\frac{1}{\sqrt{t}}+\sqrt{K}+\sqrt{\mathcal{P}}+\Gamma_a\right]\sqrt{1+\ln\frac{B}{u}}. \tag{4.3}$$

for all $(x,t)\in Q_{\frac{R}{2},T}$ with $t\neq 0$, where $A=1+\ln B-\ln\left(\inf_{Q_{R,T}}u\right)$ and

$$\Gamma_a=\sup_{Q_{R,T}}\left\{(a^{+})^{\frac{1}{2}}+|\nabla a|^{\frac{1}{3}}\right\},$$

$$\mathcal{P}=\sup_{Q_{R,T}}\left\{\left[2F'(u)-\frac{2F(u)}{u}+\frac{1}{1+\ln B-\ln u}\frac{F(u)}{u}\right]^{+}\right\}.$$

On the other hand, we can give a local gradient estimate for the positive bounded solutions to the general type of nonlinear parabolic equation (4.1) under the Yamabe flow.

Theorem 4.3. *Let $\left(M,g(x,t),e^{-f(x,t)}d\mu\right)_{t\in[0,T]}$ be a complete solution to the Yamabe flow* (1.11) *on an n-dimensional smooth manifold M and u be a smooth positive solution to the nonlinear heat equation* (4.1) *in $\mathcal{Q}_{R,T}$. Assume that $0<u\leq B$ and $\mathrm{Ric}_f\geq -(n-1)K, S\leq\mathcal{H}$ for some $K,\mathcal{H}\geq 0$ in $\mathcal{Q}_{R,T}$. Then there exists a constant c depending only n such that*

$$\frac{|\nabla u|}{u}\leq c\left[\frac{\sqrt{A}}{R}+\sqrt{\frac{\mu^{+}}{R}}+\frac{1}{\sqrt{t}}+\sqrt[4]{K^2+\mathcal{H}^2+\mathcal{P}^2}+\Gamma_a\right]\sqrt{1+\ln\frac{B}{u}} \tag{4.4}$$

for all $(x,t)\in\mathcal{Q}_{\frac{R}{2},T}$ with $t\neq 0$, where $A,\Gamma_a,\mathcal{P}$ are the same as Theorem 4.1.

Chapter 5

Rigidity and vanishing theorems for complete translating solitons

This chapter is written on the basis of the paper "Ha Tuan Dung, Nguyen Thac Dung, and Tran Quang Huy (2023), Rigidity and vanishing theorems for complete translating solitons, *Manuscripta Mathematica* Vol. 172, pp. 331-352". In this chapter, we will investigate several rigidity theorems and study the connectedness at infinity of complete translators in Euclidean spaces. This content was mentioned in Problem 4 in Chapter 1. Recall that a submanifold $X : M^n \to \mathbb{R}^{m+n}$ of the Euclidean space is said to be a translating soliton (abbreviated by translator) for the mean curvature flow if its mean curvature vector field H satisfies the equation

$$H = V^{\perp}, \tag{5.1}$$

for some fixed unit length constant vector V in $\mathbb{R}^{n+m}$, where $V^{\perp}$ is the normal projection of V to the normal bundle of $\mathbb{R}^{n+m}$.

Assume that the L^q-norm of the trace-free second fundamental form is finite, for some $q \in \mathbb{R}$ and using a Sobolev inequality, we first show that a translator in the Euclidean space $\mathbb{R}^{n+m}$ must be a linear subspace.

Theorem 5.1. *Let $X : M^{n\geq 3} \to \mathbb{R}^{m+n}$ be a smooth complete translating soliton in the Euclidean space $\mathbb{R}^{m+n}$. If the trace-free second fundamental form Φ of M satisfies*

$$\left(\int_M |\Phi|^n d\mu\right)^{\frac{1}{n}} < K(n,a) \quad \textit{and} \quad \int_M |\Phi|^{2a} e^{\langle V,X\rangle} d\mu < \infty,$$

where

$$1 \leq a < \frac{n + \sqrt{n^2 - 2n}}{2},$$

$$K(n,a)=\sqrt{\frac{(n-2)^2\left(a-\frac{1}{2}\right)}{D^2(n)\left[\frac{(n-2)^2\left(a-\frac{1}{2}\right)}{2n\left(na-\frac{n}{2}-a^2\right)}+(n-1)^2\right]\iota a^2}},\quad \iota=\begin{cases}2 & \textit{if } m=1\\ 4 & \textit{if } m\geq 2\end{cases},$$

and $D(n)$ is the Sobolev constant, then M is a linear subspace.

Applying the Sobolev inequality, we are able to obtain the following theorem.

Theorem 5.2. *Let $X: M^{n\geq 3}\to\mathbb{R}^{n+1}$ be a smooth complete translating soliton in the Euclidean space $\mathbb{R}^{n+1}$ contained in the halfspace*

$$\Pi_{V,a}=\left\{y\in\mathbb{R}^{n+1}:\langle y,V\rangle\geq a\right\},$$

for some $a\in\mathbb{R}$. If the second fundamental form A of M satisfies

$$\left(\int_M |A|^n \varrho d\mu\right)^{\frac{1}{n}}<\sqrt{\frac{(n^2-2n+2)(n-2)^2}{n^3S(n)^2(n-1)^2}},$$

where $S(n)$ is the Sobolev constant and $\rho=e^{\langle V,X\rangle}$, then M is a hyperplane.

Moreover, using the weighted Sobolev inequality, we obtain a vanishing theorem as follows.

Theorem 5.3. *Let $X: M^{n\geq 3}\to\mathbb{R}^{n+1}$ be a smooth complete translating soliton in the Euclidean space $\mathbb{R}^{n+1}$ contained in the halfspace*

$$\Pi_{V,a}=\left\{y\in\mathbb{R}^{n+1}:\langle y,V\rangle\geq a\right\},$$

for some $a\in\mathbb{R}$. Assume that for any $p\geq 2$,

$$\left(\int_M |A|^n e^{-f} d\mu\right)^{\frac{1}{n}}<\frac{\sqrt{(p-1)}(n-1)}{pS(n)},$$

where $f=-\langle X,V\rangle$ and $S(n)$ is the Sobolev constant. Then there are no non-trivial L^p_f f-harmonic 1-forms on M.

List of Author's Related Papers

1. Ha Tuan Dung, Hung Tran (2025), "On isometry groups of gradient Ricci solitons", to appear in *Forum Mathematicum* (SCI-E, Q1), https://doi.org/10.1515/forum-2024-0325.

2. Ha Tuan Dung, Nguyen Tien Manh, and Nguyen Dang Tuyen (2023), "Liouville type theorems and gradient estimates for nonlinear heat equations along ancient K-super Ricci flow via reduced geometry", *Journal of Mathematical Analysis and Applications*, Vol. 519 (2), 126836 (SCI-E, Q1).

3. Ha Tuan Dung (2023), "Gradient estimates for a general type of nonlinear parabolic equations under geometric conditions and related problems", *Nonlinear Analysis*, Vol. 226, 113135 (SCI-E, Q1).

4. Ha Tuan Dung, Nguyen Thac Dung, and Tran Quang Huy (2023), "Rigidity and vanishing theorems for complete translating solitons", *Manuscripta Mathematica*, Vol. 172, pp. 331-352 (SCI-E, Q2).