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THE TAMED-ADAPTIVE EULER-MARUYAMA SCHEME FOR SOME CLASSES OF STOCHASTIC DIFFERENTIAL EQUATIONS WITH IRREGULAR COEFFICIENTS

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INTRODUCTION

1. Reasons for choosing the topic

Stochastic analysis and stochastic differential equations were introduced by Kiyosi Itô [33] and are widely used in many application fields such as financial mathematics, physics, biology, control optimization, filtering theory,... The equations are increasingly improved to suit reality. For example, in the theory of stock pricing in mathematical finance by Black - Scholes [4] and Merton [57], the price of a stock is initially modeled by a linear stochastic differential equation. But later, it was found that this equation did not closely reflect reality and therefore the stochastic volatility model [19] was introduced. This new model describes stock prices by more complex equations with coefficients that do not satisfy the global Lipschitz condition. Equations with coefficients that do not satisfy the global Lipschitz condition also appear in many other fields.

When solving practical problems related to stochastic differential equations, we often have to calculate quantities of the form $\mathbb{E}[f(X)]$ where X is the solution of the equation which modeling the random quantity we are interested in, and f is some function. Accurate calculation of the above quantity can only be done with a very small number of equations as well as f functions. Therefore, we often have to build a suitable Monte Carlo-style scheme to approximate the calculation. Specifically, we approximate X by a quantity $X^{(n)}$ that can be simulated on a computer, where n is proportional to the number of calculations needed to determine $X^{(n)}$. Then we estimate as follows

$$\mathbb{E}[f(X)] \approx \mathbb{E}\left[f\left(X^{(n)}\right)\right] \approx \frac{1}{N}\sum_{i=1}^{N} f\left(X^{(n,i)}\right)$$

where $X^{(n,i)}$, with $i = 1, \ldots, N$, is N independent copies of $X^{(n)}$ generated on the computer. To design an optimal algorithm, that is, choose the value of n and N so that the error of the approximation does not exceed a given level with the smallest number of calculations to be performed, we need to evaluate get the error of the above two approximations. The error of the second approximation, also known as the Monte Carlo approximation, can be controlled by the central limit theorem or through measure concentration inequalities. The error of the first approximation is more difficult to determine because it depends on the regularity of the stochastic differential equation coefficients as well as smoothness of function f.

Recently, Giles [20] introduced a multilevel Monte Carlo method to approximate $\mathbb{E}[f(X)]$. This new method has a much lower computational load than the classical Monte Carlo method. One of the key points to apply the multi-level Monte Carlo method is that we must evaluate the convergence speed in the L^p space of the approximate solution to the correct solution.

When approximating the solution of a stochastic differential equation, we want the approximate solution to not only converge but also preserve the properties of the correct solution, such as stability or geometric properties of the value domain. In general, the classical Euler-Maruyama approximation scheme often does not preserve these properties. When the coefficients of the equation satisfy the local Lipschitz condition, many improved Euler-Maruyama schemes have been constructed, such as hidden, semi-hidden, projected Euler-Maruyama schemes, etc. However, the corresponding research results for equations with irregular coefficients are still quite limited.

The McKean-Vlasov stochastic differential equations have attracted a lot of attention recently. They describe a limiting behavior of individual particles interacting with each other in a mean-field manner, the so-called "Propagation of Chaos result". They are nonlinear SDEs with coefficients involving the law of the process themself. They are used across many fields such as statistical physics, neuroscience, and finance among others. This latter result also paves the way (there are others) for the numerical solution of such McKeanVlasov SDEs via approximation of the associated system of interacting particles. Though the numerical approximation in a finite time interval for McKean-Vlasov SDEs has been studied by a number of authors recently (see [32, 67] and the references therein), its approximation in infinite time intervals is still a very challenging problem. We think that this problem can be solved by using a modification of the tamed-adaptive approximation scheme introduced in [36].

For the above reasons, the graduate student and the instructors chose the research topic for the thesis as: **"The tamed-adaptive euler-maruyama scheme for some classes of stochastic differential equations with irregular coefficients"**

2. Research objectives

The main purpose of the thesis is to establish the existence and uniqueness solution theorems; propose approximation schemes for classes of stochastic differential equations, stochastic differential equations with jumps, where the drift coefficient is superlinearly increasing, locally Lipschitz continuous; The diffusion coefficient is Hölder continuous, locally Hölder continuous, determining the stable product according to the moment of the solution and approximate solution for the class of stochastic differential equations with displacement coefficients of One-sided Lipschitz with negative coefficients.

3. Research subjects

The research object of the thesis are classes of stochastic differential equations of the forms

$$dX_t = b(X_t)dt + \sigma(X_t)dW_t,\ X_0 = x_0;$$

and

$$dX_t = x_0 + b(X_t)dt + \sigma(X_t)dW_t + c\,(X_{t-})\,dZ_t,\ X_0 = x_0.$$

where the displacement coefficients $b(x)$ and diffusion coefficients $\sigma(x)$ satisfy one of the following conditions:

- $b(x)$ is locally Lipschitz and one-sided Lipschitz continuous; $\sigma(x)$ is the locally $(1/2+\alpha)$-Hölder continuous.

- $b(x)$ is locally Lipschitz and one-sided Lipschitz continuous; $\sigma(x)$ is the locally $(1/2+\alpha)$-Hölder continuous and $c(x)$ is Lipschitz continuous.

- $b(x)$ and $\sigma(x)$ are non-globally Lipschitz continuous and of super-linearly growth and $c(x)$ is Lipschitz continuous.

4. Research scope

The research scope of the thesis involves stochastic analysis, stochastic differential equations and numerical analysis. The main results of the thesis are on the quantitative and qualitative properties of exact solutions and approximated solutions of the Euler-Maruyama scheme for stochastic differential equations with irregular coefficients and for stochastic differential equations with jump.

5. Research Methods

- Analyze recent research on stochastic differential equations with irregular coefficients and approximation methods.

- Perform computer simulations to analyze, evaluate and propose new approximation algorithms.

- Participate in scientific exchange activities such as conferences and seminars to exchange, discuss and update new research methods and results in professional fields.

6. Scientific and practical significance

The results of the thesis contribute to enriching the research direction on numerical solution of some classes of stochastic differential equations. It is expected that the thesis will have a number of new contributions:

- Propose a numerical scheme that strongly converges in both finite and infinite time intervals for some class of one dimensional SDEs with locally Lipschitz continuous drift and locally Hölder continuous diffusion coefficients;

- Propose a tamed-adaptive Euler-Maruyama approximation scheme for the Lévy-driven SDEs where σ is locally Hölder continuous; σ and b are superlinear growth and c is Lipschitz continuous;

- Propose a tamed-adaptive Euler-Maruyama approximation scheme for the Lévy driven SDEs where σ and b are non-globally Lipschitz continuous and of super-linearly growth, and c is Lipschitz continuous.

The thesis can be used for reference in related research by students and scientists in the fields of probability theory and mathematical statistics and in the field of numerical analysis.

7. Thesis structure

The structure of the thesis consists of ba chapters. Chapter 1 presents an overview of previous results and introduces the results achieved in the thesis. The remaining two chapters present details for the new results of the thesis.

- **Chapter 1:** Overview
- **Chapter 2:** Tamed-adaptive Euler-Maruyama scheme for Lévy-driven SDEs with irregular coefficients
- **Chapter 3:** Tamed-adaptive Euler-Maruyama scheme for Lévy-driven McKean-Vlasov SDEs with irregular coefficients

The thesis is written based on 03 published articles.

Chương 1

OVERVIEW

In this chapter, we will summarize some previous results and new results we obtained in each problem.

1.1 Stochastic differential equation driven by Brownian motion

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions. Let $W_t = \left(W_t^1, W_t^2 \ldots, W_t^m\right)^\top$, $t \geq 0$ be an m-dimensional Brownian motion defined on the space. Let $0 \leq t_0 < T < \infty$ and x_0 be an $\mathcal{F}_{t_0}$-measurable R^d-valued random variable such that $\mathbb{E}\left[\|x_0\|^2\right] < \infty$. Consider the d-dimensional stochastic differential equation of Itô type

$$dX_t = b(t, X_t)dt + \sigma(t, X_t)dW_t \quad \text{on } t_0 \leq t \leq T \tag{1.1}$$

with initial value $x(t_0) = x_0$. By the definition of stochastic differential, this equation is equivalent to the following stochastic integral equation:

$$X(t) = x_0 + \int_{t_0}^t b(s, X(s))ds + \int_{t_0}^t \sigma(s, X(s))dW(s) \quad \text{on } t_0 \leq t \leq T. \tag{1.2}$$

Theorem 1.1.11 *Assume that there exist two positive constants K and $\overline{K}$ such that*

(i) *(Lipschitz condition)) for all $x, y \in \mathbb{R}^d$ và $t \in [t_0, T]$*

$$\|b(t,x) - b(t,y)\|^2 \vee \|\sigma(t,x) - \sigma(t,y)\|^2 \leq \overline{K}\|x-y\|^2. \tag{1.3}$$

(ii) *(Linear growth condition)) for all $x, y \in \mathbb{R}^d \times [t_0, T]$*

$$\|b(t,x)\|^2 \vee |\sigma(t,x)|^2 \leq K(1 + \|x\|^2). \tag{1.4}$$

Then there exists a unique solution $X(t)$ to equation (1.1) and the solution such that

$$\mathbb{E}\left[\int_{t_0}^T \|X_s\|^2 ds\right] < \infty. \tag{1.5}$$

Theorem 1.1.14. *Let $p \geq 2$ and $x_0 \in L^p\left(\Omega; \mathbb{R}^d\right)$. Assume that there exists a constant $\alpha > 0$ such that for all $(t,x) \in [t_0, T] \times \mathbb{R}^d$,*

$$x^\top b(t,x) + \frac{p-1}{2}\|\sigma(t,x)\|^2 \leq \alpha\left(1 + \|x\|^2\right). \tag{1.6}$$

Then

$$\mathbb{E}\left[|X_t|^p\right] \leq 2^{\frac{p-2}{2}}\left(1 + \mathbb{E}\left[|x_0|^p\right]\right)e^{p\alpha(t-t_0)} \qquad \text{for all } t \in [t_0, T]. \tag{1.7}$$

1.2 Stochastic differential equations driven by Lévy process

Proposition 1.2.23. (**Lévy-Itô decomposition**) *Let $Z = (Z_t)_{t\geq 0}$ be a Lévy process on $\mathbb{R}^d$ and ν be its Lévy measure.*

- *ν is a Radon measure on $\mathbb{R}^d_0$ and verifies:*

$$\int_{\mathbb{R}^d_0} (1 \wedge |z|^2)\nu(dz) < \infty.$$

- *The jump measure of Z, denoted by N, is a Poisson random measure on $[0, \infty] \times \mathbb{R}^d$ with intensity measure $\nu(dz)dt$.*

- *There exist a vector γ and a d-dimensional Brownian motion $(W_t)_{t\geq 0}$ with covariance matrix A such that*

$$Z_t = \gamma t + W_t + \int_0^t \int_{|z|\geq 1} zN(ds,dz) + \int_0^t \int_{|z|<1} z\widetilde{N}(ds,dz). \tag{1.8}$$

The terms in (1.8) are independent.

Note that, in the case that $Z = (Z_t)_{t\geq 0}$ is a d-dimensional centered pure jump Lévy process whose Lévy measure ν satisfies $\int_{\mathbb{R}^d}\left(1 \wedge |z|^2\right)\nu(dz) < +\infty$, the Lévy-Itô decomposition of Z is given by

$$Z_t = \int_0^t \int_{\mathbb{R}^d_0} z(N(ds,dz) - \nu(dz)ds).$$

Proposition 1.2.24. (**Lévy-Khinchin representation**) *Let $Z = (Z_t)_{t\geq 0}$ be a Lévy process on $\mathbb{R}^d$. There exists a continuous function $\psi : \mathbb{R}^d \mapsto \mathbb{R}$ called the characteristic exponent of Z, such that:*

$$\mathbb{E}\left[e^{iuZ_t}\right] = e^{t\psi(u)}, \quad u \in \mathbb{R}^d. \tag{1.9}$$

where

$$\psi(u) = i\gamma u - \frac{1}{2}u \cdot Au + \int_{\mathbb{R}^d}\left(e^{iuz} - 1 - iuz\mathbf{1}_{\{|z|\leq 1\}}\right)\nu(dz)$$

The triplet (A, ν, γ) is called *characteristic triplet* or *Lévy triplet* of the process Z_t.

Next, we now state the Burkholder-Davis-Gundy inequality for the compensated Poisson stochastic integral which will be useful in the thesis.

Lemma 1.2.25. *Let $\mathcal{B}(\mathbb{R}_0^d)$ be the Borel σ-algebra of $\mathbb{R}_0^d$ and $\mathcal{P}$ be the progressive σ-algebra on $\mathbb{R}_+ \times \Omega$. Let g be a $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}_0^d)$-measurable function satisfying that $\int_0^T \int_{\mathbb{R}_0^d} |g(s,z)|^2 \nu(dz) ds < \infty$ $\mathbb{P}$-a.s. for all $T \geq 0$. Then for any $p \geq 2$, there exists a positive constant C_p such that*

$$\begin{aligned}
&\mathbb{E}\left[\sup_{t\in[0,T]}\left|\int_0^t \int_{\mathbb{R}_0^d} g(s,z)\widetilde{N}(ds,dz)\right|^p\right] \\
&\leq C_p\left(\mathbb{E}\left[\left(\int_0^T\int_{\mathbb{R}_0^d}|g(s,z)|^2\nu(dz)ds\right)^{\frac{p}{2}}\right] + \mathbb{E}\left[\int_0^T\int_{\mathbb{R}_0^d}|g(s,z)|^p\nu(dz)ds\right]\right).
\end{aligned}$$

Furthermore, for any $1 \leq p < 2$, there exists a positive constant C_p such that

$$\mathbb{E}\left[\sup_{t\in[0,T]}\left|\int_0^t \int_{\mathbb{R}_0^d} g(s,z)\widetilde{N}(ds,dz)\right|^p\right] \leq C_p \mathbb{E}\left[\left(\int_0^T\int_{\mathbb{R}_0^d}|g(s,z)|^2\nu(dz)ds\right)^{\frac{p}{2}}\right].$$

Next, consider processes $X = (X_t)_{t\geq 0}$, admitting stochastic integral representation in the form

$$X_t = x + \int_0^t b(X_s)ds + \int_0^t \sigma(X_s)dW_s + \int_0^t \int_{\mathbb{R}_0} c(X_{s^-}, z)\widetilde{N}(ds,dz). \tag{1.10}$$

Theorem 1.2.26. The one-dimensional Itô formula. *Let $X = (X_t)_{t\geq 0}$, be the Itô-Lévy process given by (1.10) and let $f : (0,\infty) \times \mathbb{R} \to \mathbb{R}$ be a function in $C^{1,2}((0,\infty)\times\mathbb{R})$ and define*

$$Y_t := f(t, X_t), \quad t \geq 0.$$

Then the process $Y = (Y_t)_{t\geq 0}$, is also an Itô-Lévy process and its differential form is given by

$$\begin{aligned}
dY_t = &\frac{\partial f}{\partial t}(t,X_t)dt + \frac{\partial f}{\partial x}(t,X_t)b(X_t)dt + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(t,X_t)\sigma^2(X_t)dt \\
&+ \frac{\partial f}{\partial x}(t,X_t)\sigma(X_t)dW_t \\
&+ \int_{\mathbb{R}_0}\left[f(t,X_t + c(X_{t^-},z)) - f(t,X_t) - \frac{\partial f}{\partial x}(t,X_t)c(X_{t^-},z)\right]\nu(dz)dt \\
&+ \int_{\mathbb{R}_0}\left[f\left(t,X_{t^-} + c(X_{t^-},z)\right) - f\left(t,X_{t^-}\right)\right]\widetilde{N}(dt,dz).
\end{aligned} \tag{1.11}$$

Theorem 1.2.27. The multi-dimensional Itô formula. *Let $X = (X_t)_{t\geq 0}$, be an d-dimensional Itô-Lévy process. Let $f : (0,\infty)\times \mathbb{R}^d \to \mathbb{R}$ be a function in $C^{1,2}\left((0,\infty)\times\mathbb{R}^d\right)$ and define*

$$Y_t := f(t, X_t), \quad t \geq 0.$$

Then the process $Y = (Y_t)_{t\geq 0}$, is a one-dimensional Itô-Lévy process and its differential form is given by

$$
\begin{aligned}
& dY(t) \\
&= \frac{\partial f}{\partial t}(t, X_t)dt + \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(t, X_t)b_i(X_t)dt + \sum_{i=1}^{d}\sum_{j=1}^{d} \frac{\partial f}{\partial x_i}(t, X_t)\sigma_{ij}(X_t)dW_t^j \\
&+ \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{d} \frac{\partial^2 f}{\partial x_i \partial x_j}(t, X_t)\left(\sigma\sigma^T\right)_{ij}(X_t)dt + \sum_{j=1}^{d}\int_{\mathbb{R}_0}\Big[f\left(t, X_t + c^{(j)}(X_{t^-}, z)\right) \\
&- f(t, X_t) - \sum_{i=1}^{d}\frac{\partial f}{\partial x_i} f(t, X_t) c_{ij}(X_{t^-}, z)\Big]\nu_j\left(dz_j\right)dt \\
&+ \sum_{j=1}^{d}\int_{\mathbb{R}_0}\left[f\left(t, X_{t^-} + c^{(j)}(X_{t^-}, z)\right) - f\left(t, X_{t^-}\right)\right]\widetilde{N}_j\left(dt, dz_j\right), \qquad (1.12)
\end{aligned}
$$

where $c^{(j)}$ is the column number j of the $d \times d$ matrix $c = [c_{ij}]$.

Finally, we consider a process $X = (X_t)_{t\geq 0}$, the solution to the following stochastic differential equation with jumps in $\mathbb{R}^d$:

$$dX_t = b\left(X_t\right)dt + \sum_{j=1}^{d}\sigma_j\left(X_t\right)dW_t^j + \int_{\mathbb{R}_0} c\left(X_{t-}, z\right)\widetilde{N}(dt, dz),$$

with initial condition $X_0 = x_0 \in \mathbb{R}^d$. The coefficients $\sigma_j, b : \mathbb{R}^d \to \mathbb{R}^d$ and $c : \mathbb{R}^d \times \mathbb{R}_0 \to \mathbb{R}^d$ are measurable functions satisfying the following Lipschitz and linear growth conditions: for all $x, y \in \mathbb{R}^d$,

$$\max_j \left\{ |\sigma_j(x) - \sigma_j(y)|^2 ; |b(x) - b(y)|^2 ; \int_{\mathbb{R}_0} |c(x, z) - c(y, z)|^2 \nu(dz) \right\} \leq K|x - y|^2$$

and

$$\max_j \left\{ |\sigma_j(x)|^2 ; |b(x)|^2 ; \int_{\mathbb{R}_0} |c(x, z)|^2 \nu(dz) \right\} \leq K\left(1 + |x|^2\right).$$

Theorem 1.2.28. *There exists a unique càdlàg, adapted, and Markov process X on $(\Omega, \mathcal{F}, \mathbb{P})$ satisfying the integral equation*

$$X_t = x_0 + \int_0^t b\left(X_s\right)ds + \sum_{j=1}^{d}\int_0^t \sigma_j\left(X_s\right)dW_s^j + \int_0^t\int_{\mathbb{R}_0} c\left(X_{s-}, z\right)\widetilde{N}(ds, dz).$$

Moreover, for any $T > 0$,

$$\mathbb{E}\left[\sup_{t\in[0,T]} |X_t|^2\right] < \infty.$$

Chương 2

TAMED-ADAPTIVE EULER-MARUYAMA SCHEME FOR LÉVY-DRIVEN SDEs WITH IRREGULAR COEFFICIENTS

This chapter presents a new approximation scheme for a class of stochastic differential equations with irregular coefficients. We consider both the case of SDEs driven by Brownian motion (continuous noise) and the more general case of SDEs driven by Lévy processes (which include jumps). Since the Brownian-driven model is a special case of the Lévy model and the proof methods are logically parallel, this chapter will, for the sake of conciseness, present the detailed analysis only for the more general Lévy case. The corresponding results for the simpler Brownian-driven model can be inferred similarly. These results are written based on the articles [1, 2] in section **List of Author's Related Papers**.

2.1 Model assumptions

We consider the process $X = (X_t)_{t\geq 0}$ as a solution to the following stochastic differential equation (SDE) with jumps

$$X_t = x_0 + \int_0^t b(X_s)ds + \int_0^t \sigma(X_s)dW_s + \int_0^t c\left(X_{s-}\right)dZ_s. \tag{2.1}$$

The integral equation of (2.1) can be written as

$$X_t = x_0 + \int_0^t b(X_s)ds + \int_0^t \sigma(X_s)dW_s + \int_0^t \int_{\mathbb{R}_0} c\left(X_{s-}\right) z\widetilde{N}(ds, dz).$$

Assume that coefficients b, σ, c and the Lévy measure ν satisfy the following conditions:

B1. There exists a positive constant L_0 such that

$$|c(x)| \leq L_0(1 + |x|), \quad \forall x \in \mathbb{R}.$$

B2. For some $p_0 \in [2; +\infty)$, there exist constants $\gamma \in \mathbb{R}$, $\eta \in [0; +\infty)$ such that

$$xb(x) + \frac{p_0 - 1}{2}\sigma^2(x) + \frac{c^2(x)}{2L_0}\int_{\mathbb{R}_0} |z|\left((1 + L_0|z|)^{p_0-1} - 1\right)\nu(dz) \leq \gamma x^2 + \eta, \quad \forall x \in \mathbb{R}.$$

B3. Coefficient b is locally Lipschitz: for any $R>0$, there exists a positive constant L_R such that

$$|b(x)-b(y)| \leq L_R|x-y|, \quad \forall |x| \vee |y| \leq R.$$

B4. Coefficient σ is locally $\left(\alpha+\frac{1}{2}\right)$ -Hölder continuous: for any $R>0$, there exist positive constants L_R and $\alpha \in \left(0, \frac{1}{2}\right]$ such that

$$|\sigma(x)-\sigma(y)| \leq L_R|x-y|^{1/2+\alpha}, \quad \forall |x| \vee |y| \leq R.$$

B5. Coefficient c is locally Lipschitz: for any $R>0$, there exists a positive constant L_R such that

$$|c(x)-c(y)| \leq L_R|x-y|, \quad \forall |x| \vee |y| \leq R.$$

B6. $\int_{|z|>1} |z|^p \nu(dz) < \infty$ for all $p \in [1; 2p_0]$ and $\int_{0<|z|\leq 1} |z|\nu(dz) < \infty$.

B7. Coefficient b is one-sided Lipschitz: there exists a constant L_1 such that

$$(x-y)(b(x)-b(y)) \leq L_1|x-y|^2, \quad \forall x, y \in \mathbb{R}.$$

B8. Coefficient b is locally Lipschitz continuous: there exist positive constants l and L_2 such that

$$|b(x)-b(y)| \leq L_2\left(1+|x|^l+|y|^l\right)|x-y|, \quad \forall x, y \in \mathbb{R}.$$

B9. Coefficient σ is $(\alpha+\frac{1}{2})$-locally Hölder continuous: there exist positive constants m, L_3 and $\alpha \in [0, \frac{1}{2}]$ such that

$$|\sigma(x)-\sigma(y)| \leq L_3\left(1+|x|^m+|y|^m\right)|x-y|^{1/2+\alpha}, \quad \forall x, y \in \mathbb{R}.$$

B10. Coefficient c is Lipschitz: there exists a positive constant L_4 such that

$$|c(x)-c(y)| \leq L_4|x-y|, \quad \forall x, y \in \mathbb{R}.$$

2.2 Lévy-driven SDEs with irregular coefficients

Theorem 2.3.1. *Assume that the coefficients b, c and σ satisfy the conditions **B1**–**B5**. Assume further that the Lévy measure satisfies $\int_{\mathbb{R}_0} |z|\nu(dz) < \infty$ and $\int_{\mathbb{R}_0} z^2\nu(dz) < \infty$. Then the path-wise uniqueness holds for equation (2.1).*
Moreover, suppose that there exist positive constants C and $\ell \in (0; \frac{p_0}{4}]$ such that

$$|b(x)| \vee |\sigma(x)| \vee |c(x)| \leq C\left(1+|x|^\ell\right),$$

*for all $x \in \mathbb{R}$, where p_0 is defined in Condition **B2**. Then the equation (2.1) has a strong solution.*

2.3 Tamed-adaptive Euler-Maruyama scheme

For each $\Delta \in (0,1)$, the tamed-adaptive Euler-Maruyama discretisation of equation (2.1) is defined as follows

$$\begin{cases} t_0 = 0, \quad \widehat{X}_0 = x_0, \quad t_{k+1} = t_k + h(\widehat{X}_{t_k})\Delta, \\ \widehat{X}_{t_{k+1}} = \widehat{X}_{t_k} + b(\widehat{X}_{t_k})\,(t_{k+1} - t_k) + \sigma_\Delta(\widehat{X}_{t_k})(W_{t_{k+1}} - W_{t_k}) \\ \qquad\qquad + c_\Delta\left(\widehat{X}_{t_k}\right)\left(Z_{t_{k+1}} - Z_{t_k}\right), \end{cases} \tag{2.2}$$

where

$$h(x) = \frac{1}{(1 + |b(x)| + |\sigma(x)| + |x|^l)^2 + |c(x)|^{p_0}}, \tag{2.3}$$

The next result provides a sufficient condition for $t_k \to \infty$ as $k \to \infty$, which implies that the tamed adaptive approximation scheme (2.2) is well-defined.

Proposition 2.4.1. *Suppose that there exist positive constants L and β such that the coefficients $b, c, \sigma, c_\Delta, \sigma_\Delta$ satisfy the following conditions*

T1. $|b(x)| \vee |\sigma(x)| \leq L\left(1 + |x|^\beta\right)$*;*

T2. $x(b(x) - b(0)) \leq L|x|^2$*;*

T3. $|\sigma_\Delta(x)| \leq L|\sigma(x)|$ *and* $|c_\Delta(x)| \leq |c(x)|$*;*

T4. $|\sigma_\Delta(x)| \leq \frac{L}{\sqrt{\Delta}}$*;* $|c_\Delta(x)| \leq \frac{L}{\sqrt{\Delta}}$ *and* $|b(x)c_\Delta(x)| \leq \frac{L}{\sqrt{\Delta}}$*;*

for any $x \in \mathbb{R}$. Assume further that the Lévy measure satisfies $\int_{\mathbb{R}_0} z^2 \nu(dz) < \infty$. Then

$$\lim_{k \to +\infty} t_k = +\infty \quad a.s.$$

2.4 Moments

Theorem 2.5.4. *Assume that Conditions **T1**–**T4** and **B6** hold, and for some $p_0 \in [2, +\infty)$, there exist constants $\gamma \in \mathbb{R}$, $\eta \in [0, +\infty)$ such that for all $x \in \mathbb{R}$,*

$$xb(x) + \frac{p_0 - 1}{2}\sigma_\Delta^2(x) + \frac{c_\Delta^2(x)}{2L_0}\int_{\mathbb{R}_0} |z|\left((1 + L_0|z|)^{p_0 - 1} - 1\right)\nu(dz) \leq \gamma x^2 + \eta. \tag{2.4}$$

Then, for any positive integer $k \leq p_0/2$, there exists a positive constant $C = C(x_0, k, \eta, \gamma, L, L_0, p_0)$ which does not depend neither on t nor on Δ such that

$$\mathbb{E}\left[|\widehat{X}_t|^{2k}\right] \vee \mathbb{E}\left[|\widehat{X}_{\underline{t}}|^{2k}\right] \leq \begin{cases} Ce^{2k\gamma t} & \text{if } \gamma > 0, \\ C(1+t)^k & \text{if } \gamma = 0, \\ C & \text{if } \gamma < 0. \end{cases} \tag{2.5}$$

2.5 Convergence of the tamed-adaptive Euler-Maruyama scheme

Theorem 2.6.2. *Assume that Conditions **B2, B6–B10** hold and* $p_0 \geq \max\{4l; 2 + 4\alpha + 4m\}$. *Assume that the functions* $c, b, \sigma, c_\Delta, \sigma_\Delta$ *and the Lévy measure* ν *satisfy all conditions of Theorem 3.5.4, and*

$$|c(x) - c_\Delta(x)| \leq L_5 \Delta^{1/2} c^2(x)(1 + |b(x)|), \quad |\sigma(x) - \sigma_\Delta(x)| \leq L_5 \Delta^{1/2} \sigma^2(x), \tag{2.6}$$

for all $x \in \mathbb{R}$ *and some constant* $L_5 > 0$.
Then, for any $T > 0$, *there exists a positive constant* $C_T = C(x_0, L, L_0, L_1, L_2, L_3, L_4, L_5, \gamma, \eta, T)$ *such that*

$$\sup_{0 \leq t \leq T} \mathbb{E}\left[|\widehat{X}_t - X_t|\right] \leq \begin{cases} C_T \Delta^\alpha & \text{if } 0 < \alpha \leq \frac{1}{2}, \\ \dfrac{C_T}{\log \frac{1}{\Delta}} & \text{if } \alpha = 0. \end{cases} \tag{2.7}$$

Moreover, let $\mu := \int_{\mathbb{R}_0} |z| \nu(dz)$ *and assume that* $L_1 + 2L_4\mu < 0$, $\gamma < 0$, *then there exists a positive constant* $C = C(x_0, L, L_0, L_1, L_2, L_3, L_4, L_5, \gamma, \eta)$ *which does not depend on* T *such that*

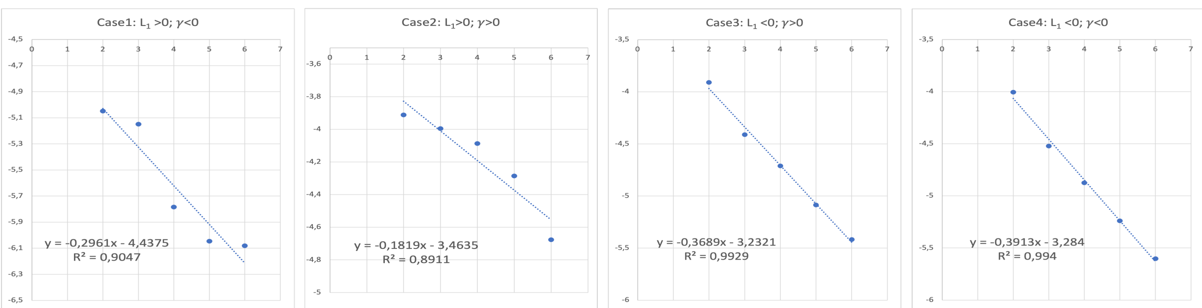
$$\sup_{t \geq 0} \mathbb{E}\left[|\widehat{X}_t - X_t|\right] \leq \begin{cases} C \Delta^\alpha & \text{if } 0 < \alpha \leq \frac{1}{2}, \\ \dfrac{C}{\log \frac{1}{\Delta}} & \text{if } \alpha = 0. \end{cases} \tag{2.8}$$

2.6 Numerical experiments

Consider numerical experiments for four SDEs with coefficients given in Table 2.1.

Case	b	σ	c	p_0	L_1	γ	η	l	m	α
1	$-1 + x - x^3$	$1 + (1 + x)x^{2/3}$	$x + \sin(x)$	10	1	-1	31873	2	$\frac{4}{3}$	$\frac{1}{6}$
2	$-1 + x - x^3$	$1 + \sqrt{\frac{x^4 + x^{4/3}}{14}}$	$x + \sin(x)$	10	1	1	957	2	2	$\frac{1}{6}$
3	$-1 - x - x^{7/3}$	$1 + \sqrt{\frac{2x^2 + x^{10/3} + x^{4/3}}{14}}$	$x + \sin(x)$	10	-1	1	1868	$\frac{4}{3}$	1	$\frac{1}{6}$
4	$-1 - x - x^{7/3}$	$1 + \sqrt{\frac{x^{10/3} + x^{4/3}}{14}}$	$x + \sin(x)$	10	-1	-1	1583	$\frac{4}{3}$	1	$\frac{1}{6}$

Bảng 2.1: Four jump SDEs with their parameters



Hình 2.1: Values of $\log_2(me(l))$ for $l = 2, 3, 4, 5, 6$.

Chương 3

TAMED-ADAPTIVE EULER-MARUYAMA SCHEME FOR LÉVY-DRIVEN MCKEAN-VLASOV SDEs WITH IRREGULAR COEFFICIENTS

Following the analysis of the tamed-adaptive Euler-Maruyama scheme for stochastic differential equations with jumps in Chapter 2, this chapter presents new results on approximating solutions for a particular class of equations, namely McKean-Vlasov SDEs with jumps. The core difference is that the coefficients of the McKean-Vlasov equation depend on both the state and the probability distribution of the process, creating a complex mean-field interaction structure. In this chapter, we focus on the case where the drift and diffusion coefficients are non-globally Lipschitz continuous and have superlinear growth. These results are written based on the article [3] in the section **List of Author's Related Papers**.

3.1 Model assumptions

On a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, we consider the d-dimensional process $X = (X_t)_{t\geq 0}$ solution to the following McKean-Vlasov stochastic differential equation (SDE) with jumps

$$dX_t = b(X_t, \mathcal{L}_{X_t})dt + \sigma(X_t, \mathcal{L}_{X_t})dW_t + c\left(X_{t-}, \mathcal{L}_{X_{t-}}\right) dZ_t, \tag{3.1}$$

for $t \geq 0$, where $X_0 = x_0 \in \mathbb{R}^d$ is a fixed initial value, $\mathcal{L}_{X_t}$ denotes the marginal law of the process X at time t.

We denote by $\mathcal{P}(\mathbb{R}^d)$ the space of all probability measures defined on a measurable space $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$, where $\mathcal{B}(\mathbb{R}^d)$ denotes the Borel σ-field over $\mathbb{R}^d$, and by

$$\mathcal{P}_2(\mathbb{R}^d) := \left\{\mu \in \mathcal{P}(\mathbb{R}^d) : \int_{\mathbb{R}^d} |x|^2 \mu(dx) < \infty\right\}$$

the subset of probability measures with finite second moment. As metric on the space $\mathcal{P}_2(\mathbb{R}^d)$, we use the $\mathcal{L}_2$-Wasserstein distance. That is, for $\mu, \nu \in \mathcal{P}_2(\mathbb{R}^d)$, the $\mathcal{L}_2$-Wasserstein distance between μ and ν is defined as

$$\mathcal{W}_2(\mu,\nu) := \inf_{\pi\in\mathcal{C}(\mu,\nu)} \left(\int_{\mathbb{R}^d\times\mathbb{R}^d} |x-y|^2 \, \pi(dx,dy) \right)^{1/2},$$

where $\mathcal{C}(\mu,\nu)$ denotes all the couplings of μ and ν. That is, $\pi \in \mathcal{C}(\mu,\nu)$ if and only if $\pi(\cdot,\mathbb{R}^d) = \mu(\cdot)$ and $\pi(\mathbb{R}^d,\cdot) = \nu(\cdot)$.

The integral equation (3.1) now writes as

$$X_t = x_0 + \int_0^t b(X_s,\mathcal{L}_{X_s})ds + \int_0^t \sigma(X_s,\mathcal{L}_{X_s})dW_s + \int_0^t \int_{\mathbb{R}_0^d} c\left(X_{s-},\mathcal{L}_{X_{s-}}\right) z\widetilde{N}(ds,dz),$$

for any $t \geq 0$.

Assume that the drift, diffusion and jump coefficients b,σ,c and the Lévy measure ν of equation (3.1) satisfy the following conditions:

C1. There exists a positive constant L such that

$$2\langle x, b(x,\mu)\rangle + |\sigma(x,\mu)|^2 + |c(x,\mu)|^2 \int_{\mathbb{R}_0^d} |z|^2\nu(dz) \leq L\left(1+|x|^2+\mathcal{W}_2^2(\mu,\delta_0)\right),$$

for any $x\in\mathbb{R}^d$ and $\mu\in\mathcal{P}_2(\mathbb{R}^d)$.

C2. There exist constants $\kappa_1 > 0, \kappa_2 > 0$, $L_1 \in \mathbb{R}$ and $L_2 \geq 0$ such that

$$\begin{aligned} &2\langle x-\overline{x}, b(x,\mu)-b(\overline{x},\overline{\mu})\rangle + \kappa_1 |\sigma(x,\mu)-\sigma(\overline{x},\overline{\mu})|^2 \\ &+\kappa_2|c(x,\mu)-c(\overline{x},\overline{\mu})|^2 \int_{\mathbb{R}_0^d} |z|^2\nu(dz) \leq L_1|x-\overline{x}|^2 + L_2\mathcal{W}_2^2(\mu,\overline{\mu}), \end{aligned}$$

for any $x,\overline{x}\in\mathbb{R}^d$ and $\mu,\overline{\mu}\in\mathcal{P}_2(\mathbb{R}^d)$.

C3. $b(x,\mu)$ is a continuous function of $x\in\mathbb{R}^d$ and $\mu\in\mathcal{P}_2(\mathbb{R}^d)$.

C4. There exist constants $L>0$ and $\ell \geq 1$ such that

$$|b(x,\mu)-b(\overline{x},\overline{\mu})| \leq L\left(1+|x|^\ell+|\overline{x}|^\ell\right)\left(|x-\overline{x}|+\mathcal{W}_2(\mu,\overline{\mu})\right),$$

for any $x,\overline{x}\in\mathbb{R}^d$ and $\mu,\overline{\mu}\in\mathcal{P}_2(\mathbb{R}^d)$.

C5. There exists an even integer $p_0 \in [2,+\infty)$ such that $\int_{|z|>1} |z|^{p_0}\nu(dz) < \infty$ and $\int_{0<|z|\leq 1} |z|\nu(dz) < \infty$.

C6. There exists a positive constant L_0 such that

$$|c(x,\mu)| \leq L_0\left(1+|x|+\mathcal{W}_2(\mu,\delta_0)\right),$$

for any $x\in\mathbb{R}^d$ and $\mu\in\mathcal{P}_2(\mathbb{R}^d)$.

C7. For the even integer $p_0 \in [2, +\infty)$ given in **C5**, there exist constants $\gamma_1 \in \mathbb{R}$, $\gamma_2 \geq 0$ and $\eta \geq 0$ such that

$$\langle x, b(x,\mu)\rangle + \frac{p_0 - 1}{2}|\sigma(x,\mu)|^2 + \frac{1}{2L_0}|c(x,\mu)|^2 \int_{\mathbb{R}_0^d} |z| \left((1 + L_0|z|)^{p_0-1} - 1\right) \nu(dz)$$
$$\leq \gamma_1|x|^2 + \gamma_2 \mathcal{W}_2^2(\mu, \delta_0) + \eta,$$

for any $x \in \mathbb{R}^d$ and $\mu \in \mathcal{P}_2(\mathbb{R}^d)$.

3.2 Lévy-driven McKean-Vlasov SDEs with irregular coefficients

We first recall a result on the existence and uniqueness for strong solution of McKean-Vlasov SDE with jumps (3.1).

Proposition 3.3.1. *Assume Conditions **C1**, **C3** and that Condition **C2** holds for* $\kappa_1 = \kappa_2 = 1, L_1 = L_2 > 0$. *Then, there exists a unique càdlàg process* $X = (X_t)_{t\geq 0}$ *taking values in* $\mathbb{R}^d$ *satisfying the McKean-Vlasov SDE with jumps (3.1) such that*

$$\sup_{t\in[0,T]} \mathbb{E}\left[|X_t|^2\right] \leq K,$$

where $T > 0$ *is a fixed constant and* $K := K(|x_0|^2, d, L, L_1, T)$ *is a positive constant.*

We next show the following moment estimates of exact solution $X = (X_t)_{t\geq 0}$.

Proposition 3.3.2. *Let* $X = (X_t)_{t\geq 0}$ *be a solution to equation (3.1). Assume Conditions **C6**, **C7**, and that* σ *is bounded on* $C \times \mathcal{P}_2(\mathbb{R}^d)$ *for every compact subset* C *of* $\mathbb{R}^d$, *and **C5** holds for* $q = 2p_0$. *Then for any* $p \in [2, p_0]$, *there exists a positive constant* C_p *such that for any* $t \geq 0$

$$\mathbb{E}\left[|X_t|^p\right] \leq \begin{cases} C_p(1 + e^{\gamma p t}) & \text{if} \quad \gamma \neq 0, \\ C_p(1+t)^{p/2} & \text{if} \quad \gamma = 0, p = 2 \quad \text{or} \quad \gamma = 0, \gamma_2 > 0, p \in (2, p_0], \\ C_p(1+t)^p & \text{if} \quad \gamma = 0, \gamma_2 = 0, p \in (2, p_0], \end{cases} \tag{3.2}$$

where $\gamma = \gamma_1 + \gamma_2$.

Note that if $\gamma < 0$, we have that $\sup_{t\geq 0} \mathbb{E}\left[|X_t|^p\right] \leq 2C_p$.

3.3 Propagation of chaos

We now consider the system of non-interacting particles which is associated with the Lévy-driven McKean-Vlasov SDE (3.1), where the state $X^i = (X_t^i)_{t\geq 0}$ of the particle i is defined by

$$X_t^i = x_0 + \int_0^t b(X_s^i, \mathcal{L}_{X_s^i})ds + \int_0^t \sigma(X_s^i, \mathcal{L}_{X_s^i})dW_s^i + \int_0^t c\left(X_{s-}^i, \mathcal{L}_{X_{s-}^i}\right) dZ_s^i$$

$$= x_0 + \int_0^t b(X_s^i, \mathcal{L}_{X_s^i})ds + \int_0^t \sigma(X_s^i, \mathcal{L}_{X_s^i})dW_s^i$$
$$+ \int_0^t \int_{\mathbb{R}_0^d} c\left(X_{s-}^i, \mathcal{L}_{X_{s-}^i}\right) z\widetilde{N}^i(ds, dz), \tag{3.3}$$

for any $t \geq 0$ and $i \in \{1, \dots, N\}$.

For $\boldsymbol{x}^N := (x_1, x_2, \dots, x_N), \boldsymbol{y}^N := (y_1, y_2, \dots, y_N) \in \mathbb{R}^{dN}$, we have

$$\mathcal{W}_2^2(\mu^{\boldsymbol{x}^N}, \delta_0) = \frac{1}{N}\sum_{i=1}^N |x_i|^2 .$$

Here, the empirical measure is defined by $\mu^{\boldsymbol{x}^N}(dx) := \frac{1}{N}\sum_{i=1}^N \delta_{x_i}(dx)$, where δ_x denotes the Dirac measure at x. Moreover, a standard bound for the Wasserstein distance between two empirical measures $\mu^{\boldsymbol{x}^N}, \mu^{\boldsymbol{y}^N}$ is given by

$$\mathcal{W}_2^2(\mu^{\boldsymbol{x}^N}, \mu^{\boldsymbol{y}^N}) \leq \frac{1}{N}\sum_{i=1}^N |x_i - y_i|^2 = \frac{1}{N}\left|\boldsymbol{x}^N - \boldsymbol{y}^N\right|^2 ,$$

Now, the true measure $\mathcal{L}_{X_t}$ at each time t is approximated by the empirical measure

$$\mu_t^{\boldsymbol{X}^N}(dx) := \frac{1}{N}\sum_{i=1}^N \delta_{X_t^{i,N}}(dx), \tag{3.4}$$

where $\boldsymbol{X}^N = (\boldsymbol{X}_t^N)_{t\geq 0} = (X_t^{1,N}, \dots, X_t^{N,N})_{t\geq 0}^{\intercal}$, which is called the system of interacting particles, is the solution to the $\mathbb{R}^{dN}$-dimensional Lévy-driven SDE with components $X^{i,N} = (X_t^{i,N})_{t\geq 0}$

$$X_t^{i,N} = x_0 + \int_0^t b(X_s^{i,N}, \mu_s^{\boldsymbol{X}^N})ds + \int_0^t \sigma(X_s^{i,N}, \mu_s^{\boldsymbol{X}^N})dW_s^i + \int_0^t c\left(X_{s-}^{i,N}, \mu_{s-}^{\boldsymbol{X}^N}\right) dZ_s^i$$
$$= x_0 + \int_0^t b(X_s^{i,N}, \mu_s^{\boldsymbol{X}^N})ds + \int_0^t \sigma(X_s^{i,N}, \mu_s^{\boldsymbol{X}^N})dW_s^i$$
$$+ \int_0^t \int_{\mathbb{R}_0^d} c\left(X_{s-}^{i,N}, \mu_{s-}^{\boldsymbol{X}^N}\right) z\widetilde{N}^i(ds, dz), \tag{3.5}$$

for any $t \geq 0$ and $i \in \{1, \dots, N\}$.

Proposition 3.4.1. *Let $X^{i,N} = (X_t^{i,N})_{t\geq 0}$ be a solution to equation (3.5). Assume Conditions **C6, C7** and that σ is bounded on $C \times \mathcal{P}_2(\mathbb{R}^d)$ for every compact subset C of $\mathbb{R}^d$, and **C5** holds for $q = 2p_0$. Then for any $p \in [2, p_0]$, there exists a positive constant C_p such that for any $t \geq 0$,*

$$\max_{i\in\{1,\dots,N\}} \mathbb{E}\left[|X_t^{i,N}|^p\right] \leq \begin{cases} C_p(1 + e^{\gamma p t}) & \textit{if} \quad \gamma \neq 0, \\ C_p(1 + t)^{p/2} & \textit{if} \quad \gamma = 0, p = 2 \quad \textit{or} \quad \gamma = 0, \gamma_2 > 0, p \in (2, p_0], \\ C_p(1 + t)^p & \textit{if} \quad \gamma = 0, \gamma_2 = 0, p \in (2, p_0], \end{cases}$$

where $\gamma = \gamma_1 + \gamma_2$.

Note that when $\gamma < 0$, we have that $\max_{i\in\{1,\dots,N\}} \sup_{t\geq 0} \mathbb{E}\left[|X_t^{i,N}|^p\right] \leq 2C_p$.

Proposition 3.4.2. *Assume that all conditions in Proposition 3.4.1 hold and that Condition **C2** holds for $\kappa_1 = \kappa_2 = 1$, $L_1 \in \mathbb{R}$, $L_2 \geq 0$. Then, we have*

$$\max_{i\in\{1,\dots,N\}} \sup_{t\in[0,T]} \mathbb{E}\left[\left|X_t^i - X_t^{i,N}\right|^2\right] \leq C_T \varphi(N), \tag{3.6}$$

for any $N \in \mathbb{N}$, where the positive constant C_T does not depend on N.
Assume further that $L_1 + L_2 < 0$ and $\gamma < 0$. Then, we have

$$\max_{i\in\{1,\dots,N\}} \sup_{t\geq 0} \mathbb{E}\left[\left|X_t^i - X_t^{i,N}\right|^2\right] \leq C\varphi(N), \tag{3.7}$$

for any $N \in \mathbb{N}$, where the positive constant C does not depend on N and T.

3.4 Tamed-adaptive Euler-Maruyama scheme

Let $\sigma_\Delta = (\sigma_{\Delta,ij})_{1\leq i,j\leq d} : \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}^d \otimes \mathbb{R}^d$ and $c_\Delta = (c_{\Delta,ij})_{1\leq i,j\leq d} : \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}^d \otimes \mathbb{R}^d$ be approximations of the coefficients σ and c, respectively, which will be specified later. For all $i \in \{1,\dots,N\}$, $\Delta \in (0,1)$ and $k \in \mathbb{N}$, we define the tamed-adaptive Euler-Maruyama discretization of equation (3.5) by

$$\begin{cases} t_0 = 0, \quad \widehat{X}_0^{i,N} = x_0, \quad t_{k+1} = t_k + \mathbf{h}(\widehat{\boldsymbol{X}}_{t_k}^N, \mu_{t_k}^{\widehat{\boldsymbol{X}}^N})\Delta, \\ \widehat{X}_{t_{k+1}}^{i,N} = \widehat{X}_{t_k}^{i,N} + b(\widehat{X}_{t_k}^{i,N}, \mu_{t_k}^{\widehat{\boldsymbol{X}}^N})(t_{k+1} - t_k) + \sigma_\Delta(\widehat{X}_{t_k}^{i,N}, \mu_{t_k}^{\widehat{\boldsymbol{X}}^N})(W_{t_{k+1}}^i - W_{t_k}^i) \\ \qquad\qquad + c_\Delta(\widehat{X}_{t_k}^{i,N}, \mu_{t_k}^{\widehat{\boldsymbol{X}}^N})(Z_{t_{k+1}}^i - Z_{t_k}^i), \end{cases} \tag{3.8}$$

where

$$\widehat{\boldsymbol{X}}_{t_k}^N = \left(\widehat{X}_{t_k}^{1,N}, \dots, \widehat{X}_{t_k}^{N,N}\right),$$
$$\mu_{t_k}^{\widehat{\boldsymbol{X}}^N}(dx) := \frac{1}{N}\sum_{i=1}^N \delta_{\widehat{X}_{t_k}^{i,N}}(dx),$$
$$\mathbf{h}(\widehat{\boldsymbol{X}}_{t_k}^N, \mu_{t_k}^{\widehat{\boldsymbol{X}}^N}) = \min\left\{h(\widehat{X}_{t_k}^{1,N}, \mu_{t_k}^{\widehat{\boldsymbol{X}}^N}), \dots, h(\widehat{X}_{t_k}^{N,N}, \mu_{t_k}^{\widehat{\boldsymbol{X}}^N})\right\},$$

and

$$h(x,\mu) = \frac{h_0}{(1 + |b(x,\mu)| + |\sigma(x,\mu)| + |x|^\ell)^2 + |c(x,\mu)|^{p_0}}, \tag{3.9}$$

for $x \in \mathbb{R}^d$, $\mu \in \mathcal{P}_2(\mathbb{R}^d)$ and some positive constant h_0. Here, the constants ℓ and p_0 are respectively defined in Conditions **C4** and **C7**.

Now, we provide sufficient conditions to ensure $t_k \uparrow \infty$ as $k \uparrow \infty$, which shows that the tamed-adaptive Euler-Maruyama approximation scheme (3.8) is well-defined.

Proposition 3.5.1. *Assume that Condition **C5** holds for $p = 2$ and there exist positive constants L, β_1 and β_2 such that the functions h, b, σ_Δ and c_Δ satisfy*

T1. $\dfrac{1}{h(x,\mu)} \le L\left(1+|x|^{\beta_1}+\mathcal{W}_2^{\beta_2}(\mu,\delta_0)\right)$; $|b(x,\mu)|\,(1+|b(x,\mu)|)\,h(x,\mu)\le L$;

T2. $\langle x, b(x,\mu)-b(0,\delta_0)\rangle \le L\left(|x|^2+\mathcal{W}_2^2(\mu,\delta_0)\right)$;

T3. $|\sigma_\Delta(x,\mu)|\,(1+|x|)\le \dfrac{L}{\sqrt{\Delta}}$; $|c_\Delta(x,\mu)|\,(1+|x|)\le \dfrac{L}{\sqrt{\Delta}}$;

$|b(x,\mu)||c_\Delta(x,\mu)|\le \dfrac{L}{\sqrt{\Delta}}$;

for any $x\in\mathbb{R}^d$ *and* $\mu\in\mathcal{P}_2(\mathbb{R}^d)$. *Then, we have*

$$\lim_{k\to+\infty} t_k = +\infty \quad a.s.$$

3.5 Moments

Firstly, we are going to show that the moments of $\widehat{X}_t^{i,N}$ depend on t. For this, we need to introduce the following condition.

T4. There exists a positive constant L such that $|\sigma_\Delta(x,\mu)|\le|\sigma(x,\mu)|$ and $|c_\Delta(x,\mu)|\le |c(x,\mu)|$ for all $x\in\mathbb{R}^d$ and $\mu\in\mathcal{P}_2(\mathbb{R}^d)$;

T5. For some integer $p_0\in[2,+\infty)$, there exist constants $\widetilde{L_0}>0$, $\widetilde{\gamma}_1\in\mathbb{R}$, $\widetilde{\gamma}_2>0$, $\widetilde{\eta}\ge 0$ such that for all $x\in\mathbb{R}^d$ and $\mu\in\mathcal{P}_2(\mathbb{R}^d)$,

$$|c_\Delta(x,\mu)|\le \widetilde{L_0}\left(1+|x|+\mathcal{W}_2(\mu,\delta_0)\right), \tag{3.10}$$

and

$$\begin{aligned}&\langle x,b(x,\mu)\rangle+\frac{p_0-1}{2}|\sigma_\Delta(x,\mu)|^2+|c_\Delta(x,\mu)|^2\int_{\mathbb{R}_0^d}\left[\frac{|z|^2}{2}+\frac{1}{\widetilde{L_0}^2}\right.\\&\times\left.\left(\left(1+|z|(\widetilde{L_0}+\epsilon)\right)^{p_0-1}-1-|z|(\widetilde{L_0}+\epsilon)\right)\left(|z|\Big(\frac{\widetilde{L_0}}{2}+\epsilon\Big)+\epsilon\right)\right]\nu(dz)\\&\le\widetilde{\gamma}_1|x|^2+\widetilde{\gamma}_2\mathcal{W}_2^2(\mu,\delta_0)+\widetilde{\eta},\end{aligned} \tag{3.11}$$

where $\epsilon=\frac{1}{2\sqrt{N}}\max\{3\widetilde{L_0},1\}$.

Proposition 3.6.9. *Assume Conditions* ***T1–T5*** *and* ***C5****. Then, for any positive* $k\le p_0/2$*, there exists a positive constant* $C=C(x_0,k,\widetilde{\gamma}_1,\widetilde{\gamma}_2,\widetilde{\eta},L,\widetilde{L_0},p_0)$ *which depends neither on* Δ *nor on* t *such that for any* $t\ge0$,

$$\max_{i\in\{1,\dots,N\}}\left(\mathbb{E}\left[|\widehat{X}_t^{i,N}|^{2k}\right]\vee\mathbb{E}\left[|\widehat{X}_{\underline{t}}^{i,N}|^{2k}\right]\right)\le\begin{cases}Ce^{2k\widetilde{\gamma}t} & if\quad \widetilde{\gamma}>0,\\ C(1+t)^k & if\quad \widetilde{\gamma}=0,\\ C & if\quad \widetilde{\gamma}<0,\end{cases} \tag{3.12}$$

where $\widetilde{\gamma}=\widetilde{\gamma}_1+\widetilde{\gamma}_2$.

3.6 Convergence

Firstly, the following additional condition will be needed.

T6. There exists a positive constant L_3 such that

$$|\sigma(x,\mu)-\sigma_\Delta(x,\mu)| \le L_3\Delta^{1/2}|\sigma(x,\mu)|^2(1+|x|),$$
$$|c(x,\mu)-c_\Delta(x,\mu)| \le L_3\Delta^{1/2}|c(x,\mu)|^2\left(1+|x|+|b(x,\mu)|\right),$$

for all $x\in\mathbb{R}^d$ and $\mu\in\mathcal{P}_2(\mathbb{R}^d)$.

Remark 3.7.1. If we choose

$$\sigma_\Delta(x,\mu) = \frac{\sigma(x,\mu)}{1+\Delta^{1/2}|\sigma(x,\mu)|(1+|x|)}, \tag{3.13}$$
$$c_\Delta(x,\mu) = \frac{c(x,\mu)}{1+\Delta^{1/2}|c(x,\mu)|(1+|x|+|b(x,\mu)|)}, \tag{3.14}$$

then Conditions **T3, T4** and **T6** are satisfied.

Theorem 3.7.2. *Assume Conditions **C1**, **C3–C5**, **T2–T6**, and $p_0 \ge 4\ell+6$. Assume further that there exists a constant $\varepsilon>0$ such that **C2** holds for $\kappa_1=\kappa_2=1+\varepsilon$, $L_1\in\mathbb{R}$, $L_2\ge 0$. Then for any $T>0$, there exist positive constants $C_T = C(x_0,L,L_1,L_2,L_3,\widetilde{\gamma}_1,\widetilde{\gamma}_2,\widetilde{\eta},\widetilde{L_0},T)$ and $C'_T = C'(x_0,L,L_1,L_2,L_3,\widetilde{\gamma}_1,\widetilde{\gamma}_2,\widetilde{\eta},\widetilde{L_0},T)$ such that*

$$\max_{i\in\{1,\dots,N\}}\sup_{t\in[0,T]}\mathbb{E}\left[\left|X_t^{i,N}-\widehat{X}_t^{i,N}\right|^2\right] \le C_T\Delta, \tag{3.15}$$

and for any $p\in(0,2)$,

$$\max_{i\in\{1,\dots,N\}}\mathbb{E}\left[\sup_{t\in[0,T]}\left|X_t^{i,N}-\widehat{X}_t^{i,N}\right|^p\right] \le \left(\frac{4-p}{2-p}\right)(C'_T\Delta)^{p/2}. \tag{3.16}$$

Moreover, if $\widetilde{\gamma}=\widetilde{\gamma}_1+\widetilde{\gamma}_2<0$, and $L_1+L_2<0$, then, there exists a positive constant $C''=C''(x_0,L,L_1,L_2,L_3,\widetilde{\gamma}_1,\widetilde{\gamma}_2,\widetilde{\eta},\widetilde{L_0})$, which does not depend on T, such that

$$\max_{i\in\{1,\dots,N\}}\sup_{t\ge 0}\mathbb{E}\left[\left|X_t^{i,N}-\widehat{X}_t^{i,N}\right|^2\right] \le C''\Delta. \tag{3.17}$$

Theorem 3.7.3. *Assume Conditions **C1**, **C3–C7**, **T2–T6**, and $p_0 \ge 4\ell+6$. Assume further that there exists a constant $\varepsilon>0$ such that **C2** holds for $\kappa_1=\kappa_2=1+\varepsilon$, $L_1\in\mathbb{R}$, $L_2\ge 0$. Then for any $T>0$, there exists a positive constant $C_T = C(x_0,L,L_1,L_2,L_3,\widetilde{\gamma}_1,\widetilde{\gamma}_2,\widetilde{\eta},\widetilde{L_0},T)$ such that*

$$\max_{i\in\{1,\dots,N\}}\sup_{t\in[0,T]}\mathbb{E}\left[\left|X_t^{i}-\widehat{X}_t^{i,N}\right|^2\right] \le C_T\left(\Delta+\varphi(N)\right), \tag{3.18}$$

for any $N\in\mathbb{N}$, where the constant $C_T>0$ does not depend on N.
Moreover, assume that $\gamma=\gamma_1+\gamma_2<0$, $\widetilde{\gamma}=\widetilde{\gamma}_1+\widetilde{\gamma}_2<0$ and $L_1+L_2<0$. Then, there

exists a positive constant
$C'' = C''(x_0, L, L_1, L_2, L_3, \widetilde{\gamma}_1, \widetilde{\gamma}_2, \widetilde{\eta}, \widetilde{L_0})$ *which does not depend on* T *such that*

$$\max_{i\in\{1,...,N\}} \sup_{t\geq 0} \mathbb{E}\left[\left|X_t^i - \widehat{X}_t^{i,N}\right|^2\right] \leq C''\left(\Delta + \varphi(N)\right). \tag{3.19}$$

3.7 Numerical experiments

In this section, we consider the rate of convergence of the tamed-adaptive Euler-Maruyama scheme (3.8),(3.9), (3.13), (3.14) in Theorem 4.7.3 for fixed large values of N. We consider the following Lévy-driven McKean-Vlasov stochastic differential equation

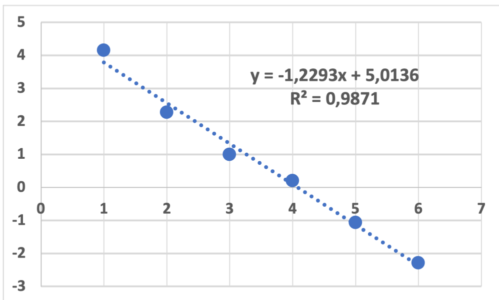
$$\begin{aligned} dX_t = &\left(-1 - 3\left(X_t + \mathbb{E}\left[X_t\right]\right) - X_t|X_t|^{0.3}\right) dt + 0.2\left(1 + |X_t|^{1.1} + \mathbb{E}\left[X_t\right]\right) dW_t \\ &+ 0.2\left(X_{t-} + \mathbb{E}\left[X_{t-}\right]\right) dZ_t. \end{aligned} \tag{3.20}$$

That is,

$$b(x,\mu) = -1 - 3\left(x + \int_{\mathbb{R}} z\mu(dz)\right) - x|x|^{0.3},$$
$$\sigma(x,\mu) = 0.2\left(1 + |x|^{1.1} + \int_{\mathbb{R}} z\mu(dz)\right), \qquad c(x,\mu) = 0.2\left(x + \int_{\mathbb{R}} z\mu(dz)\right),$$

for all $x \in \mathbb{R}$ and $\mu \in \mathcal{P}_2(\mathbb{R})$. Here we suppose that $Z = (Z_t)_{t\geq 0}$ is a bilateral Gamma process whose scale parameter is 5 and shape parameter is 1. It is straightforward to verify that these coefficients satisfy Conditions **A1**–**A7** and **T2**–**T6**.

Figure 3.1 shows the values of $\log_2 MSE(\mathbf{l}, T)$ plotted against $\mathbf{l} \in \{1, 2, ..., 6\}$. We see that $\beta \approx 0.5$.



Hình 3.1: Error $\log_2 \mathrm{MSE}(\mathbf{l}, 10)$ plotted against $\mathbf{l} = 1, ..., 6$.

CONCLUSIONS AND RECOMMENDATIONS

Conclusions

By combining modern tools in stochastic analysis, stochastic differential equations, numerical analysis and approximation techniques of Yamada-Watanabe, the thesis has built appropriate approximation schemes for a number of classes of stochastic differential equations with irregular coefficients. The main results of the thesis are on the quantitative and qualitative properties of exact solutions and approximated solutions of the Euler-Maruyama scheme for stochastic differential equations with irregular coefficients and for stochastic differential equations with jump.

The main results of the thesis include:

1) Propose a tamed-adaptive Euler-Maruyama approximation scheme that strongly converges in both finite and infinite time intervals for some class of one-dimensional SDEs with locally Lipschitz continuous drift and locally Hölder continuous diffusion coefficients.

2) Propose a tamed-adaptive Euler-Maruyama approximation scheme that strongly converges in both finite and infinite time intervals for the Lévy-driven SDEs where b is locally Lipschitz continuous; σ is locally Hölder continuous and c is Lipschitz continuous.

3) Propose a tamed-adaptive Euler-Maruyama approximation scheme that strongly converges in both finite and infinite time intervals for the Lévy-driven McKean-Vlasov SDEs where b and σ are non-globally Lipschitz continuous and super-linearly growth, and c is Lipschitz continuous.

Recommendations

In the process of researching the problems of the thesis, we thought about some further research directions as follows:

- The approximation methods preserve the geometrical and asymptotic properties of a system of random differential equations with complex structures, such as a system of non-coliding random points or a positive system.
- Weak convergence of tamed-adaptive approximation chemes.
- Construct approximation schemes with faster convergence for stochastic differential equations with smooth and super-linear growth coefficients.

LIST OF AUTHOR'S RELATED PAPERS

[1] Kieu T.T., Luong D.T., Ngo H.L. (2022), "Tamed-adaptive Euler-Maruyama approximation for SDEs with locally Lipschitz continuous drift and locally Hölder continuous diffusion coefficients", *Stoch Anal Appl* 40(4), pp. 714-734.

[2] Kieu T.T., Luong D.T., Ngo H.L., Tran N.K. (2022), "Strong convergence in infinite time interval of tamed-adaptive Euler-Maruyama scheme for Lévy-driven SDEs with irregular coefficients", *Comp. Appl. Math.* 41, 301.

[3] Tran N.K., Kieu T.T., Luong D.T., Ngo H.L. (2025), "On the infinite time horizon approximation for Lévy-driven McKean-Vlasov SDEs with non-globally Lipschitz continuous and super-linearly growth drift and diffusion coefficients", *J. Math. Anal. Appl.*, 54, paper no. 128982, 38 pp.