VIETNAM NATIONAL UNIVERSITY, HANOI
VNU UNIVERSITY OF SCIENCE

KIEU TRUNG THUY

THE TAMED-ADAPTIVE EULER-MARUYAMA SCHEME FOR SOME CLASSES OF STOCHASTIC DIFFERENTIAL EQUATIONS WITH IRREGULAR COEFFICIENTS

DOCTORAL THESIS IN MATHEMATICS

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CONFIRMATION

This thesis was written on the basis of my research carried out at VNU University of Science under the supervision of Assoc. Prof. Dr. Ngo Hoang Long and Prof. Dr. Nguyen Huu Du. The results presented in this thesis are my own original work and have not been submitted, in whole or in part, for any other degree.

The author

Kieu Trung Thuy

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The author

Kieu Trung Thuy

CONTENTS

LIST OF NOTATION

Throughout the thesis, we agree on a number of symbols as follows:

a.s	Almost surely
càdlàg	Right-continuous functions with left-hand limits
i.i.d	Independently and identically distributed
$A := B$	A is defined by B or A is denoted by B
$a \vee b$	The maximum of a and b
$a \wedge b$	The minimum of a and b
$\mathbb{R}_+$	The set of all nonnegative real numbers
$\mathbb{R}^d$	The d-dimensional Euclidean space
$\mathbb{R}^d_0$	$:= \mathbb{R}^d \backslash \{0\}$
$\mathbb{R}^{d\times m}$	The space of real $d \times m-$matrices
$\lvert x\rvert$	The Euclidean norm of a vector x
A^T	The transpose of a vector or matrix A
trace(A)	The trace of a square matrix $A = (a_{ij})_{d\times d}$, i.e. $\operatorname{trace} A = \sum_{1\le i\le d} a_{ii}$
$\lvert A\rvert$	$= \sqrt{\operatorname{trace}(A^T A)}$, i.e. the trace norm of a matrix A
$(\Omega, \mathcal{F}, \mathbb{P})$	The complete probability space with a filtration $\{\mathcal{F}_t\}_{t\ge 0}$
$\mathbb{E}[X]$	The expectation of random variable X
limsup	The upper limit
$C[a;b]$	The family of continuous functions defined on $[a;b]$
$C^{1,2}(\mathbb{R}_+ \times \mathbb{R}; \mathbb{R})$	The family of all real-valued functions $V(t,x)$ defined on $\mathbb{R}_+ \times \mathbb{R}$ which are continuously twice differentiable in $x \in \mathbb{R}$ and once differentiable in $t \in \mathbb{R}_+$.
$\mathcal{M}^p\left([a,b]; \mathbb{R}^d\right)$	The family of processes $\{f(t)\}_{a\le t\le b}$ in $\mathcal{L}^p\left([a,b]; \mathbb{R}^d\right)$ such that $\mathbb{E}\left[\int_a^b \lvert f(t)\rvert^p dt\right] < \infty$
$\mathcal{L}^p\left([a,b]; \mathbb{R}^d\right)$	The family of R^d-valued $\mathcal{F}_t$-adapted processes $\{f(t)\}_{a\le t\le b}$ such that $\int_a^b \lvert f(t)\rvert^p dt < \infty$ a.s.
$\mathcal{L}^p\left(\mathbb{R}_+; \mathbb{R}^d\right)$	The family of processes $\{f(t)\}_{t\ge 0}$ such that for every $T > 0$, $\{f(t)\}_{0\le t\le T} \in \mathcal{L}^p\left([0;T]; R^d\right)$
$L^p(\Omega; \mathbb{R}^d)$	The family of R^d-valued random variables ξ with $\mathbb{E}[\lvert\xi\rvert^p] < \infty$

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INTRODUCTION

1. Background and Motivation

The Japanese mathematician Gisiro Maruyama is credited as the first to formally and generally propose the pioneering extension of the Euler method from ordinary differential equations to stochastic differential equations. In [64], he introduced the idea of approximating the solution of a stochastic differential equation by a sequence of simpler processes. This work led to the development of one of the most widely used approximation schemes today: the Euler-Maruyama method. Its theoretical importance was later solidified by Kloeden and Platen [46], who proved that this method has a strong convergence order of $\frac{1}{2}$ in the L^2-norm, provided that the coefficients of the equation satisfy the global Lipschitz condition.

This global Lipschitz condition is a critical assumption, and it fails to hold in many important classes of equations. When it is not satisfied, research has focused on several important classes of SDEs with ”irregular coefficients”, such as when the coefficients are **non-smooth** (e.g., merely Hölder continuous), exhibit **superlinear growth**, or are **singular**.

The challenge posed by superlinearly growing coefficients is particularly severe. For the second class of problems, involving superlinear coefficients, Hutzenthaler, Jentzen, and Kloeden [34] studied the stochastic differential equation

$$dX_t = \left(-\frac{1}{2}\bar{\sigma}^2 X_t + X_t^3\right) dt + \bar{\sigma} X_t dW_t, \quad \text{with} \quad X_0 = 1,$$

where the drift coefficient grows superlinearly. They demonstrated that while the moments of the true solution remain finite, the moments of the Euler-Maruyama approximation diverge to infinity. This divergence demonstrates that the Euler-Maruyama scheme fails to converge in the L^p-sense for any $p \geq 1$, highlighting the need for modified numerical schemes.

To overcome this divergence issue, a variety of explicit "tamed" methods have been proposed. Notable examples include the tamed Euler-Maruyama scheme, introduced by Hutzenthaler, Jentzen, and Kloeden [35] and the truncated Euler-Maruyama scheme, proposed by Mao [62]. The fundamental idea behind these "tamed" methods is to replace the original, unbounded coefficients with suitably constructed bounded functions in the numerical scheme. However, these works introduce their own limitation: they are primarily applicable to SDEs whose diffusion coefficients are (at least) locally Lipschitz.

A separate challenge arises from non-smooth coefficients. Hairer, Hutzenthaler, and Jentzen [26] constructed a notable example of a stochastic differential equation where the EM approximation fails to converge even for a bounded and infinitely differentiable (C^∞) drift because its derivatives exhibit extremely high oscillations. For the Cox-Ingersoll-Ross process, where the diffusion coefficient is merely Hölder continuous of order $\frac{1}{2}$, Hefter and Jentzen [27] showed that the convergence rate of any discrete-time approximation can deteriorate significantly. While Gyöngy and Rásonyi [25] later proved the EM scheme can achieve convergence for Hölder continuous diffusion, their results are only applicable to equations whose coefficients have at most linear growth.

This reveals a critical research gap: while "tamed" methods address superlinear growth, they struggle with non-Lipschitz diffusion, and while methods exist for non-Lipschitz diffusion, they often fail for superlinear growth.

A distinct, yet equally significant, limitation of the aforementioned works is their focus on the convergence of numerical schemes over a finite time horizon $[0;T]$. However, the long-term behavior of solutions is of paramount importance in fields such as control theory and optimization. This has motivated research into numerical approximation over infinite time intervals. For instance, Fang and Giles [18] introduced an "adaptive" Euler-Maruyama scheme and proved its convergence in the L^p-norm over an infinite time horizon. Crucially, their analysis still required the diffusion coefficient to be globally Lipschitz continuous.

This focus on long-term behavior is inextricably linked to the fundamental problem of stability. The long-term stability of a system — for instance, the survival or extinction of a species in mathematical biology, is a paramount property that a numerical method must replicate. Foundational results can be found in classical texts such as Khasminskii [42] and Mao [61]. A key challenge is that standard explicit methods like the Euler-Maruyama or Milstein schemes often fail to preserve the stability of the true solution. This shortcoming has spurred the development of alternative methods, such as implicit θ-Euler-Maruyama schemes and various tamed Euler methods [28, 32, 63, 83, 90]. Nevertheless, research on stability-preserving schemes for equations with non-regular coefficients — for example, where the drift or diffusion is merely locally Hölder continuous, remains comparatively limited.

These compelling reasons, which highlight a clear convergence of unsolved challenges, motivate the research topic of this thesis: **"The tamed-adaptive Euler-Maruyama scheme for some classes of stochastic differential equations with irregular coefficients"**. This topic is positioned to address the intersection of these gaps: combining "tamed" (for superlinearity) and "adaptive" (for long-term stability) approaches to successfully handle "irregular" (e.g., Hölder continuous) coefficients.

2. Research objectives and tasks

The primary objectives of this thesis are:

- To establish existence and uniqueness theorems for solutions to certain classes of stochastic differential equations, including those with jumps.
- To propose and analyze novel numerical approximation schemes for these equations, particularly under conditions where the drift coefficient has superlinear growth and is only locally Lipschitz continuous and the diffusion coefficient is merely Hölder continuous (or locally Hölder continuous).
- To investigate the long-term moment stability of both the true solution and its numerical approximation for stochastic differential equations whose drift coefficient satisfies a one-sided Lipschitz condition with a negative constant.

3. Research subjects

The research subjects of the thesis are classes of stochastic differential equations of the forms

$$dX_t = b(X_t)dt + \sigma(X_t)dW_t,\ X_0 = x_0;$$

and

$$dX_t = b(X_t)dt + \sigma(X_t)dW_t + c\left(X_{t-}\right)dZ_t,\ X_0 = x_0.$$

where the drift coefficients $b(x)$ and diffusion coefficients $\sigma(x)$ satisfy one of the following conditions:

- $b(x)$ is locally Lipschitz and one-sided Lipschitz continuous; $\sigma(x)$ is locally $(1/2+\alpha)$-Hölder continuous.
- $b(x)$ is locally Lipschitz and one-sided Lipschitz continuous; $\sigma(x)$ is locally $(1/2+\alpha)$-Hölder continuous and $c(x)$ is Lipschitz continuous.
- $b(x)$ and $\sigma(x)$ are non-globally Lipschitz continuous and of superlinear growth and $c(x)$ is Lipschitz continuous.

4. Research scope

The research scope of this thesis encompasses stochastic analysis, stochastic differential equations, and numerical analysis. The primary contributions of this work focus on the quantitative and qualitative properties of both exact and approximate solutions, specifically concerning the Euler-Maruyama scheme for stochastic differential equations with irregular coefficients and for stochastic differential equations with jumps.

5. Research methods

- Conduct a thorough analysis of contemporary research on stochastic differential equations with irregular coefficients and associated approximation methods.
- Execute computer simulations to analyze, evaluate, and propose innovative approximation algorithms.
- Engage in scientific exchange initiatives, including conferences and seminars, to facilitate the exchange, discussion, and updating of cutting-edge research methodologies and findings within the relevant fields.

6. Contributions of the thesis

The results of this thesis contribute to the expansion of research in the numerical solutions of specific classes of stochastic differential equations. It is anticipated that this thesis will yield several novel contributions:

- Propose a numerical scheme that achieves strong convergence over both finite and infinite time intervals for a class of one-dimensional stochastic differential equations with locally Lipschitz continuous drift and locally Hölder continuous diffusion coefficients.
- Develop a tamed-adaptive Euler-Maruyama approximation scheme for Lévy-driven stochastic differential equations where coefficient σ is locally Hölder continuous, coefficients σ and b exhibit superlinear growth, and coefficient c is Lipschitz continuous.
- Introduce a tamed-adaptive Euler-Maruyama approximation scheme for Lévy-driven McKean-Vlasov stochastic differential equations where coefficients σ and b are non-globally Lipschitz continuous, superlinearly growing, and coefficient c is Lipschitz continuous.

The thesis can be used for reference in related research by students and scientists in the fields of probability theory and mathematical statistics and in the field of numerical analysis.

7. Structure of the thesis

The thesis includes an introduction, three main chapters, conclusions, list of published works and references:

- **Chapter 1: Overview**
  In this chapter, we give the definition of stochastic differential equations; sufficient conditions for the equation to exist and have unique solutions. We also introduce sufficient conditions for the solution to satisfy a number of properties such as bounded moments, continuity over time, and asymptotic stability. We also briefly introduce some basic knowledge for stochastic differential equations with jumps such as: compound Poisson processes, Lévy processes, stochastic calculus for Lévy processes and stochastic differential equations driven by Lévy processes. At the conclusion of this chapter, we present a selection of Euler-Maruyama approximation schemes that have been previously introduced by researchers in earlier publications.

- **Chapter 2: Tamed-adaptive Euler-Maruyama scheme for Lévy-driven SDEs with irregular coefficients**
  In this chapter, a tamed-adaptive Euler-Maruyama approximation scheme is proposed for Lévy-driven stochastic differential equations with locally Lipschitz continuous, polynomial growth drift, and locally Hölder continuous, polynomial growth diffusion coefficients. We show that the new scheme converges in both finite and infinite time intervals under some suitable conditions on the regularity and the growth of the coefficients.

- **Chapter 3: Tamed-adaptive Euler-Maruyama scheme for Lévy-driven McKean-Vlasov SDEs with irregular coefficients**
  In this chapter, we study the numerical approximation for McKean-Vlasov stochastic differential equations driven by Lévy processes. We propose a tamed-adaptive Euler-Maruyama scheme and consider its strong convergence in both finite and infinite time horizons when applied for some classes of Lévy-driven McKean-Vlasov stochastic differential equations with non-globally Lipschitz continuous and superlinear growth drift and diffusion coefficients.

Chapter 1

OVERVIEW

1.1 Stochastic differential equation driven by Brownian motion

In this section, we briefly introduce some basic properties of the Itô stochastic integral used in the thesis. Symbols, definitions and properties in this section are referenced from [61].

1.1.1 Itô stochastic integral

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions. Let $(W_t)_{t\geq 0}$ be a one-dimensional Brownian motion defined on the probability space, adapted to the filtration $(\mathcal{F}_t)_{t\geq 0}$.

Theorem 1.1.1 ([61], Theorem 5.8)**.** *Let $f, g \in \mathcal{M}^2([a,b];\mathbb{R})$ and let α, β be two real numbers. Then*

(i) $\int_a^b f(t)dW_t$ *is* $\mathcal{F}_b$*-measurable;*

(ii) $\mathbb{E}\left[\int_a^b f(t)dW_t\right] = 0;$

(iii) $\mathbb{E}\left[\left|\int_a^b f(t)dW_t\right|^2\right] = \mathbb{E}\left[\int_a^b |f(t)|^2 dt\right];$

(iv) $\int_a^b [\alpha f(t) + \beta g(t)]dW_t = \alpha \int_a^b f(t)dW_t + \beta \int_a^b g(t)dW_t.$

Theorem 1.1.2 ([61], Theorem 5.9)**.** *Let $f \in \mathcal{M}^2([a,b];\mathbb{R})$. Then*

$$\mathbb{E}\left[\int_a^b f(t)dW_t|\mathcal{F}_a\right] = 0,$$

$$\mathbb{E}\left[\left|\int_a^b f(t)dW_t\right|^2 |\mathcal{F}_a\right] = \mathbb{E}\left[\int_a^b |f(t)|^2 dt|\mathcal{F}_a\right] = \int_a^b \mathbb{E}\left[|f(t)|^2|\mathcal{F}_a\right] dt.$$

Theorem 1.1.3 ([61], Theorem 5.12)**.** *Let* $f \in \mathcal{M}^2([0,T];\mathbb{R})$*. Define*

$$I(t) = \int_0^t f(s)dW_s \quad \text{for } \ 0 \le t \le T,$$

then $(I(t), \mathcal{F}_t)_{t\in[0,T]}$ *is a square-integrable martingale with respect to the filtration* $\{\mathcal{F}_t\}$*. In particular,*

$$\mathbb{E}\left[\sup_{0\le t\le T}\left|\int_0^t f(s)dW_s\right|^2\right] \le 4\mathbb{E}\left[\int_0^T |f(s)|^2 ds\right].$$

Theorem 1.1.4 ([61], Theorem 5.16, Theorem 5.17)**.** *Let* $f \in \mathcal{M}^2([0;T];\mathbb{R})$ *and let* ρ, τ *be two stopping times such that* $0 \le \rho \le \tau \le T$*. Define*

$$\int_0^\tau f(s)dW_s = \int_0^T \mathbb{I}_{[0,\tau]}(s)f(s)dW_s,$$

then

(i) $\mathbb{E}\left[\int_\rho^\tau f(s)dW_s\right] = 0$*;*

(ii) $\mathbb{E}\left[\left|\int_\rho^\tau f(s)dW_s\right|^2\right] = \mathbb{E}\left[\int_\rho^\tau |f(s)|^2 ds\right]$*;*

(iii) $\mathbb{E}\left[\left|\int_\rho^\tau f(s)dW_s\right|^2 \Big|\mathcal{F}_\rho\right] = \mathbb{E}\left[\int_\rho^\tau |f(s)|^2 ds\Big|\mathcal{F}_\rho\right]$.

Theorem 1.1.5 (***The one-dimensional Itô formula***, [61], Theorem 6.2)**.** *Let* $X(t)$ *be an Itô process on* $t \ge 0$ *with the stochastic differential*

$$dX_t = f(t)dt + g(t)dW_t,$$

where $f \in \mathcal{L}^1(\mathbb{R}_+;\mathbb{R})$ *and* $g \in \mathcal{L}^2(\mathbb{R}_+;\mathbb{R})$*. Let* $V(t,x) \in C^{1,2}(\mathbb{R}_+ \times \mathbb{R};\mathbb{R})$*. Then* $V(t,X_t)$ *is again an Itô process with the stochastic differential given by*

$$\begin{aligned} dV(t,X_t) = &\left[V_t(t,X_t) + V_x(t,X_t)f(t) + \frac{1}{2}V_{xx}(t,X_t)g^2(t)\right]dt \\ &+ V_x(t,X_t)g(t)dW_t \quad a.s. \end{aligned}$$

Theorem 1.1.6 (***The multi-dimensional Itô formula***, [61], Theorem 6.4)**.** *Let* X_t *be a d-dimensional Itô process on* $t \ge 0$ *with the stochastic differential*

$$dX_t = f(t)dt + g(t)dW_t,$$

where $f \in \mathcal{L}^1\left(\mathbb{R}_+;\mathbb{R}^d\right)$ *and* $g \in \mathcal{L}^2\left(\mathbb{R}_+;\mathbb{R}^{d\times m}\right)$*. Let* $V(t,x) \in C^{1,2}\left(\mathbb{R}_+ \times \mathbb{R}^d;\mathbb{R}\right)$*. Then* $V(t,X_t)$ *is again an Itô process with the stochastic differential given by*

$$\begin{aligned} dV(t,X_t) = &\left[V_t(t,X_t) + V_x(t,X_t)f(t) + \frac{1}{2}\operatorname{trace}\left(g^\top(t)V_{xx}(t,X_t)g(t)\right)\right]dt \\ &+ V_x(t,X_t)g(t)dW_t \quad a.s. \end{aligned}$$

Theorem 1.1.7 ([61], Theorem 7.1). *Let $p \geq 2$. Let $g \in \mathcal{M}^2\left([0;T];\mathbb{R}^{d\times m}\right)$ such that*

$$\mathbb{E}\left[\int_0^T |g(s)|^p ds\right] < \infty.$$

Then

$$\mathbb{E}\left[\left|\int_0^T g(s)dW_s\right|^p\right] \leq \left(\frac{p(p-1)}{2}\right)^{\frac{p}{2}} T^{\frac{p-2}{2}}\mathbb{E}\left[\int_0^T |g(s)|^p ds\right].$$

In particular, for $p = 2$, there is equality.

Theorem 1.1.8 (***Bürkholder-Davis-Gundy inequality***, [61], Theorem 7.3). *Let $g \in \mathcal{L}^2\left(\mathbb{R}_+;\mathbb{R}^{d\times m}\right)$. Define, for $t \geq 0$,*

$$X_t = \int_0^t g(s)dW_s \quad \text{and} \quad A_t = \int_0^t |g(s)|^2 ds.$$

Then for every $p > 0$, there exist universal positive constants c_p, C_p (depending only on p), such that

$$c_p\mathbb{E}\left[|A_t|^{\frac{p}{2}}\right] \leq \mathbb{E}\left[\sup_{0\leq s\leq t}|X_s|^p\right] \leq C_p\mathbb{E}\left[|A_t|^{\frac{p}{2}}\right]$$

for all $t \geq 0$. In particular, one may take

$$\begin{aligned}
&c_p = (p/2)^p,\ C_p = (32/p)^{p/2} && \text{if } 0 < p < 2;\\
&c_p = 1,\ C_p = 4 && \text{if } p = 2;\\
&c_p = (2p)^{-p/2},\ C_p = \left[p^{p+1}/2(p-1)^{p-1}\right]^{p/2} && \text{if } p > 2.
\end{aligned}$$

Theorem 1.1.9 (***Gronwall's inequality***, [61], Theorem 8.1). *Let $T > 0$ and $c \geq 0$. Let $u(\cdot)$ be a Borel measurable bounded nonnegative function on $[0;T]$, and let $v(\cdot)$ be a nonnegative integrable function on $[0;T]$. If*

$$u(t) \leq c + \int_0^t v(s)u(s)ds \quad \text{for all } 0 \leq t \leq T,$$

then

$$u(t) \leq c\exp\left(\int_0^t v(s)ds\right) \quad \text{for all } 0 \leq t \leq T. \tag{1.1}$$

1.1.2 Stochastic differential equations

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions. Let $W_t = \left(W_t^1, W_t^2, \ldots, W_t^m\right)^\top$, $t \geq 0$ be an $m-$dimensional Brownian motion defined on the space. Let $0 \leq t_0 < T < \infty$ and X_0 be an $\mathcal{F}_{t_0}-$measurable

$\mathbb{R}^d-$valued random variable such that $\mathbb{E}\left[|X_0|^2\right] < \infty$. Consider the $d-$dimensional stochastic differential equation of Itô type

$$dX_t = b(t, X_t)dt + \sigma(t, X_t)dW_t \quad \text{on } t_0 \le t \le T \tag{1.2}$$

with initial value $X(t_0) = X_0$. By the definition of stochastic differential, this equation is equivalent to the following stochastic integral equation:

$$X(t) = X_0 + \int_{t_0}^{t} b(s, X(s))ds + \int_{t_0}^{t} \sigma(s, X(s))dW(s) \quad \text{on } t_0 \le t \le T. \tag{1.3}$$

Definition 1.1.10 ([61], Definition 2.1)**.** *An $\mathbb{R}^d-$valued stochastic process $\{X(t)\}_{t_0 \le t \le T}$ is called a solution of equation (1.2) if it has the following properties:*

(i) $\{X_t\}$ is continuous and $\mathcal{F}_t$-adapted;

(ii) $\{b(t, X_t)\} \in \mathcal{L}^1\left([t_0, T]; \mathbb{R}^d\right)$ and $\{\sigma(t, X_t)\} \in \mathcal{L}^2\left([t_0, T]; \mathbb{R}^{d\times m}\right)$;

(iii) equation (1.3) holds for every $t \in [t_0, T]$ with probability 1.

A solution $\{X_t\}$ is said to be unique if any other solution $\{\overline{X}_t\}$ is indistinguishable from $\{X_t\}$, that is

$$\mathbb{P}\left[X_t = \overline{X}_t \text{ for all } t_0 \le t \le T\right] = 1.$$

Remark: Denote the solution of equation (1.2) by $X(t; t_0, x_0)$. From equation (1.2) that for any $s \in [t_0, T]$,

$$X_t = X_s + \int_{s}^{t} b(u, X_u)du + \int_{s}^{t} \sigma(u, X_u)dW_u \text{ on } s \le t \le T. \tag{1.4}$$

However, (1.4) is also a stochastic differential equation on $[s, T]$ with initial value $X_s = X(s; t_0, x_0)$, whose solution is denoted by $X(t; s, X(s; t_0, x_0))$. Therefore, we see that the solution of equation (1.2) satisfies the following *flow or semigroup* property

$$X(t; t_0, x_0) = X(t; s, x(s; t_0, x_0)), \quad t_0 \le s \le t \le T.$$

Theorem 1.1.11 ([61], Theorem 3.1)**.** *Assume that there exist two positive constants K and $\overline{K}$ such that*

(i) *(Lipschitz condition) for all $x, y \in \mathbb{R}^d$ and $t \in [t_0, T]$,*

$$|b(t, x) - b(t, y)|^2 \vee |\sigma(t, x) - \sigma(t, y)|^2 \le \overline{K}|x - y|^2. \tag{1.5}$$

(ii) *(Linear growth condition) for all* $(t, x) \in [t_0, T] \times \mathbb{R}^d$,

$$|b(t,x)|^2 \vee |\sigma(t,x)|^2 \leq K(1+|x|^2). \tag{1.6}$$

Then there exists a unique solution $X(t)$ *to equation* (1.2) *and the solution such that*

$$\mathbb{E}\left[\int_{t_0}^{T} |X_s|^2 ds\right] < \infty. \tag{1.7}$$

When the coefficients of the stochastic differential equation satisfy the Lipschitz condition (1.5) and the linear growth condition (1.6), we say the equation (1.2) has *regular coefficient*. When at least one of the above two conditions is not satisfied, we say that the equation (1.2) has *irregular coefficients*.

Lemma 1.1.12 ([61], Lemma 3.2)**.** *Assume that the linear growth condition* (1.6) *holds and* X_t *is a solution of equation* (1.2)*, then*

$$\mathbb{E}\left[\sup_{t\in[t_0,T]} |X_t|^2\right] \leq \left(1+3\mathbb{E}[|x_0|^2]\right)e^{3K(T-t_0)(T-t_0+4)}.$$

In particular, X_t *satisfies condition* (1.7)*.*

Consider a stochastic differential equation on $[t_0, \infty)$,

$$dX_t = b(t,X_t)dt + \sigma(t,X_t)dW_t, \quad t \in [t_0,\infty). \tag{1.8}$$

Theorem 1.1.13 ([61], Theorem 3.6)**.** *Assume that for every real number* $T > t_0$ *and integer* $n \geq 1$*, there exists a positive constant* $K_{T,n}$ *such that for all* $t \in [t_0, T]$ *and all* $x, y \in \mathbb{R}^d$ *with* $|x| \vee |y| \leq n$*,*

$$|b(t,x)-b(t,y)|^2 \vee |\sigma(t,x)-\sigma(t,y)|^2 \leq K_{T,n}|x-y|^2.$$

Assume also that for every $T > t_0$*, there exists a positive constant* K_T *such that for all* $(t,x) \in [t_0,T] \times \mathbb{R}^d$*,*

$$x^\top b(t,x) + \frac{1}{2}|\sigma(t,x)|^2 \leq K_T(1+|x|^2).$$

Then there exists a unique global solution $X(t)$ *to equation* (1.8) *and the solution belongs to* $\mathcal{M}^2\left([t_0,\infty);\mathbb{R}^d\right)$*.*

1.1.3 Boundedness and continuity of moment of solution

Assume that X_t is the unique solution of equation (1.8).

Theorem 1.1.14 ([61], Theorem 4.1, page 59). *Let $p \geq 2$ and $x_0 \in L^p\left(\Omega;\mathbb{R}^d\right)$. Assume that there exists a constant $\alpha > 0$ such that for all $(t,x) \in [t_0,T] \times \mathbb{R}^d$,*

$$x^\top b(t,x) + \frac{p-1}{2}|\sigma(t,x)|^2 \leq \alpha\left(1+|x|^2\right). \tag{1.9}$$

Then

$$\mathbb{E}\left[|X_t|^p\right] \leq 2^{\frac{p-2}{2}}\left(1+\mathbb{E}\left[|x_0|^p\right]\right)e^{p\alpha(t-t_0)} \quad \text{for all } t \in [t_0,T]. \tag{1.10}$$

If the linear growth condition (1.6) holds, then the condition (1.9) holds with $\alpha = \sqrt{K} + K(p-1)/2$ due to

$$x^\top b(t,x) \leq \frac{\sqrt{K}}{2}|x|^2 + \frac{1}{2\sqrt{K}}|b(x,t)|^2 \leq \sqrt{K}(1+|x|^2),$$

and

$$|\sigma(t,x)|^2 \leq K(1+|x|^2).$$

Theorem 1.1.15 ([61], Theorem 4.3, page 61). *Let $p \geq 2$ and $x_0 \in L^p\left(\Omega;\mathbb{R}^d\right)$. Assume that the linear growth condition (1.6) holds. Then*

$$\mathbb{E}\left[|X_t - X_s|^p\right] \leq C(t-s)^{\frac{p}{2}} \quad \text{for all } t_0 \leq s < t \leq T, \tag{1.11}$$

where

$$C = 2^{p-2}\left(1+\mathbb{E}\left[|x_0|^p\right]\right)e^{p\alpha(T-t_0)}\left(\left[2\left(T-t_0\right)\right]^{\frac{p}{2}} + \left[p(p-1)\right]^{\frac{p}{2}}\right)$$

and $\alpha = \sqrt{K} + K(p-1)/2$. In particular, the pth moment of the solution is continuous on $[t_0,T]$.

1.2 Stochastic differential equations driven by Lévy processes

In this section, we briefly introduce some basic knowledge for stochastic differential equations with jumps used in the thesis. Symbols, definitions and properties in this section are referenced from [2, 11, 14, 48, 73].

1.2.1 Poisson processes

Definition 1.2.1 ([48], Definition 2.1.17). *Let $T = (T_n)_{n\in\mathbb{N}}$ be a discrete time stochastic process on $(\Omega,\mathcal{F},\mathbb{P})$. Then T is called a point process on $\mathbb{R}^+$ if*

$$0 < T_1 < \cdots < T_n < \cdots \text{ and } T_n \uparrow \infty.$$

Definition 1.2.2 ([48], Definition 2.1.18). *$(N_t)_{t\geq 0}$ is called a counting process of the point process $T=(T_n)_{n\in\mathbb{N}}$ if*

$$N_t = \sum_{n\geq 1} \mathbf{1}_{\{T_n\leq t\}}.$$

Definition 1.2.3 ([48], Definition 2.1.20). *Let $(\tau_i)_{i\geq 1}$ be a sequence of independent exponential random variables with positive parameter λ and $T_n = \sum_{i=1}^{n} \tau_i$. The process $(N_t)_{t\geq 0}$ defined by*

$$N_t = \sum_{n\geq 1} \mathbf{1}_{\{T_n\leq t\}}$$

is called a Poisson process with intensity λ.

Proposition 1.2.4 ([48], Proposition 2.1.24). *The Poisson process $(N_t)_{t\geq 0}$ has independent increments. That is, for any partition $0<t_1<\cdots<t_n$, the random variables $N_{t_1}, N_{t_2}-N_{t_1}, \ldots, N_{t_n}-N_{t_{n-1}}$ are independent.*

Lemma 1.2.5 ([48], Lemma 2.1.25). *Let $(N_t)_{t\geq 0}$ be a counting process of the point process $(T_n)_{n\in\mathbb{N}}$ with stationary independent increments. That is:*

(i) *for all $n\geq 2, 0\leq t_0<t_1<\cdots<t_n$, the increments $\left\{N_{t_j}-N_{t_{j-1}}; 1\leq j\leq n\right\}$ are mutually independent,*

(ii) *for all $0\leq s<t$, the law of N_t-N_s depends upon the pair (s,t) only through the difference $t-s$.*

We define that $\tau_1 := T_1, \tau_i := T_i - T_{i-1}$ $(i\geq 2)$. If, for any $t\geq 0, N_t$ follows a Poisson distribution with parameter λt $(\lambda>0)$, then $(\tau_i)_{i\in\mathbb{N}}$ is a sequence of independent exponential random variables with parameter λ.

Definition 1.2.6. *Let $(N_t)_{t\geq 0}$ be a Poisson process with intensity λ. The process $(\widetilde{N}_t)_{t\geq 0}$ defined by*

$$\widetilde{N}_t = N_t - \lambda t$$

is called a compensated Poisson process. Here, $(\lambda t)_{t\geq 0}$ is called the compensator of $(N_t)_{t\geq 0}$.

Like the Poisson process, $\widetilde{N}$ also has independent increments and it is easy to show that $\left(\widetilde{N}_t\right)$ has the *martingale property:*

$$\forall t>s, \mathbb{E}\left[\widetilde{N}_t \mid \widetilde{N}_s\right] = \widetilde{N}_s. \tag{1.12}$$

1.2.2 Poisson random measures

Definition 1.2.7 ([11], Definition 2.18)**.** *Let $E \subset \mathbb{R}^d$ and a given (positive) Radon measure μ on $(E, \mathcal{E})$. A Poisson random measure on E with intensity measure μ is an integer valued random measure:*

$$\begin{aligned} M : \Omega \times \mathcal{E} &\to \mathbb{N} \\ (\omega, A) &\mapsto M(\omega, A), \end{aligned}$$

such that

(i) *For almost all $\omega \in \Omega, M(\omega, \cdot)$ is an integer-valued Radon measure on E: for any bounded measurable $A \subset E, M(A) < \infty$ is an integer−valued random variable.*

(ii) *For each measurable set $A \subset E$, $M(\cdot, A) = M(A)$ is a Poisson random variable with parameter $\mu(A)$:*

$$\forall k \in \mathbb{N}, \quad \mathbb{P}[M(A) = k] = e^{-\mu(A)} \frac{(\mu(A))^k}{k!}.$$

(iii) *For disjoint measurable sets $A_1, \ldots, A_n \in \mathcal{E}$, the variables $M(A_1), \ldots, M(A_n)$ are independent.*

The following result allows to construct, given any Radon measure μ, a Poisson random measure with intensity μ:

Proposition 1.2.8 ([11], Proposition 2.14)**.** *For any Radon measure μ on $E \subset \mathbb{R}^d$, there exists a Poisson random measure M on E with intensity μ.*

In the same way, one can construct *the compensated Poisson random measure* $\widetilde{M}$ by subtracting from M its intensity measure:

$$\widetilde{M}(A) = M(A) - \mu(A). \tag{1.13}$$

1.2.3 Compound Poisson processes

Definition 1.2.9 ([11], Definition 3.3)**.** *A compound Poisson process with intensity $\lambda > 0$ and jump size distribution F is a stochastic process $X = (X_t)_{t \geq 0}$ defined as*

$$X_t = \sum_{i=1}^{N_t} Y_i, \tag{1.14}$$

where jump sizes Y_i are i.i.d. with distribution F and $(N_t)_{t \geq 0}$ is a Poisson process with intensity λ, independent from $(Y_i)_{i \geq 1}$.

Note that, the following identity may be useful in order to prove various properties:

$$\sum_{i=1}^{N_t} Y_i = \sum_{i=1}^{\infty} Y_i \mathbf{1}_{\{T_i \le t\}}.$$

Proposition 1.2.10 ([11], Proposition 3.3). *$(X_t)_{t\geq 0}$ is compound Poisson process if and only if it is a Lévy process and its sample paths are piecewise constant functions.*

Definition 1.2.11 ([48], Definition 3.3.1). *For any $a \in \mathbb{R}^d$, the Dirac point mass measure δ_a at a is defined by:*

$$\delta_a(A) = \begin{cases} 1 \text{ if } a \in A, \\ 0 \text{ if } a \notin A, \end{cases} \tag{1.15}$$

for any $A \in \mathscr{B}\left(\mathbb{R}^d\right)$.

Definition 1.2.12 ([48], Definition 3.3.2). *Let $\left(X_t = \sum_{i=1}^{\infty} Y_i \mathbf{1}_{\{T_i \le t\}}\right)_{t\geq 0}$ be a compound Poisson process. Then N is called a Poisson random measure associated with $(X_t)_{t\geq 0}$, if*

$$N(F) := \sum_{i=1}^{\infty} \delta_{T_i} \times \delta_{Y_i}(F), \quad F \in \mathscr{B}[0,\infty) \times \mathscr{B}(\mathbb{R}^d). \tag{1.16}$$

Its differential form is denoted by $N(dt, dz)$ for any $t > 0$, $z \in \mathbb{R}_0^d$.

Proposition 1.2.13 ([48], Proposition 3.3.5). *Let N be a Poisson random measure associated with $(X_t)_{t\geq 0}$. Then*

(i) $\left(\int_0^t \int_{\mathbb{R}^d} N(ds, dz)\right)_{t\geq 0}$ *is a Poisson process.*

(ii) *For any $A \in \mathscr{B}[0,\infty), B \in \mathscr{B}(\mathbb{R}^d)$ and for any $g \in L^1([0,\infty) \times \mathbb{R}^d, \mathscr{B}[0,\infty) \times \mathscr{B}(\mathbb{R}^d), N)$,*

$$\int_A \int_B g(s,z) N(ds,dz) = \sum_{i=1}^{\infty} g\left(T_i, Y_i\right) \mathbf{1}_{\{T_i \in A\}} \mathbf{1}_{\{Y_i \in B\}}.$$

In particular, if $g : \mathbb{R}^d \to \mathbb{R}^d$, then

$$\int_0^t \int_{\mathbb{R}^d} g(z) N(ds, dz)$$

is a compound Poisson process with each jump size following the same distribution as that of $g(Y)$.

Notice that if $g(s,z)=z$ then

$$X_t = \int_0^t \int_{\mathbb{R}^d} zN(ds,dz).$$

Definition 1.2.14 ([48], Definition 3.4.1)**.** *Let $(X_t)_{t\geq 0}$ be a compound Poisson process.The jump size of X at time t $(t>0)$ is defined by*

$$\Delta X_t := X_t - X_{t^-} = X_t - \lim_{s\uparrow t} X_s,$$

and $\Delta X_0 = 0$.

Note that ΔX satisfies the following properties:

(i) For any time interval $[0;T]$, $\Delta X_s \neq 0$ for a finite number of values of s.

(ii) From the above, we may use the sum notation to have that

$$X_t = \sum_{0\leq s\leq t} \Delta X_s := \sum_{0\leq s\leq t} \Delta X_s \mathbf{1}_{\{\Delta X_s \neq 0\}}.$$

Definition 1.2.15 ([48], Definition 3.4.10)**. (Compensator, compensated Poisson random measure)** *Let N be a Poisson random measure. Then we define the compensator of N as the σ-finite measure*

$$\widehat{N}(F) := \lambda \int_F f(z)dsdz, \quad F \in \mathscr{B}[0,\infty)\otimes\mathscr{B}(\mathbb{R}^d)$$

and the compensated Poisson random measure as

$$\widetilde{N} := N - \widehat{N}.$$

Theorem 1.2.16 ([48], Theorem 3.4.11)**.** *Let g be a bounded $\mathscr{B}[0,\infty)\otimes\mathscr{B}(\mathbb{R}^d)$-measurable function where $g(\cdot,z)$ is a càdlàg function for any $z\in\mathbb{R}^d$. Then for any partition: $0\leq t_0 < t_1 < \cdots < t_n = t$,*

$$\mathbb{E}\left[\sum_{i=0}^{n-1}\int_{t_i}^{t_{i+1}}\int_{\mathbb{R}^d} g\left(t_i,z\right)N(ds,dz) \mid \mathscr{F}_{t_0}\right] = \sum_{i=0}^{n-1}\int_{t_i}^{t_{i+1}}\int_{\mathbb{R}^d} g\left(t_i,z\right)\widehat{N}(ds,dz).$$

Furthermore, if $\max\left\{\left|t_{i+1}-t_i\right|; i=0,\dots,n-1\right\}\to 0$ as $n\to\infty$ then by taking limits we also have

$$(LHS) \to \mathbb{E}\left[\int_{t_0}^t \int_{\mathbb{R}^d} g(s^-,z)N(ds,dz)\mid \mathscr{F}_{t_0}\right](n\to\infty),$$
$$(RHS) \to \int_{t_0}^t \int_{\mathbb{R}^d} g(s^-,z)\widehat{N}(ds,dz)(n\to\infty).$$

Moreover, if $\int_{t_0}^t \int_{\mathbb{R}^d} g(s^-, z)N(ds, dz) \in L^1(\Omega, \mathscr{F}, \mathbb{P})$*, then we get*

$$\mathbb{E}\left[\int_{t_0}^t \int_{\mathbb{R}^d} g(s^-, z)N(ds, dz) \mid \mathscr{F}_{t_0}\right] = \int_{t_0}^t \int_{\mathbb{R}^d} g(s^-, z)\widehat{N}(ds, dz).$$

1.2.4 Lévy processes

Definition 1.2.17 ([11], Definition 3.1)**.** *A càdlàg stochastic process* $Z = (Z_t)_{t\geq 0}$ *on* $(\Omega, \mathcal{F}, \mathbb{P})$ *with values in* $\mathbb{R}^d$ *such that* $Z_0 = 0$ *is called a Lévy process if it possesses the following properties:*

(i) *Independent increments: for every increasing sequence of times* $0 \leq t_0 < t_1 < \ldots < t_n$*, the random variables* $Z_{t_0}, Z_{t_1} - Z_{t_0}, \ldots, Z_{t_n} - Z_{t_{n-1}}$ *are independent.*

(ii) *Stationary increments: the law of* $Z_{t+h} - Z_t$ *does not depend on* t*.*

(iii) *Stochastic continuity:* $\forall \varepsilon > 0, \lim_{h\to 0} \mathbb{P}\left[|Z_{t+h} - Z_t| \geq \varepsilon\right] = 0$*.*

Definition 1.2.18 ([11], Definition 3.4)**.** *Let* $(Z_t)_{t\geq 0}$ *be a Lévy process on* $\mathbb{R}^d$*. The measure* ν *on* $\mathbb{R}^d$ *defined by:*

$$\nu(A) = \mathbb{E}\left[\sum_{t\in[0,1]} \mathbf{1}_{\{\Delta Z_t \neq 0, \Delta Z_t \in A\}}\right], \quad A \in \mathscr{B}\left(\mathbb{R}^d\right) \tag{1.17}$$

is called the Lévy measure of Z : $\nu(A)$ *is the expected number, per unit time, of jumps whose size belongs to* A*.*

Definition 1.2.19. *Let* $Z = (Z_t)_{t\geq 0}$ *be a Lévy process on* $\mathbb{R}^d$ *and* ν *be its Lévy measure. The jump measure* N *is a Poisson random measure on* $[0, \infty) \times \mathbb{R}^d$ *with intensity measure* $\nu(dz)dt$*. That is,*

$$N(dt, dz) := \sum_{0\leq s\leq t} \mathbf{1}_{\{\Delta Z_s \neq 0\}} \delta_{(s, \Delta Z_s)}(ds, dz).$$

The compensated jump measure $\widetilde{N}$*, also compensated Poisson random measure, is defined by*

$$\widetilde{N}(dt, dz) := N(dt, dz) - \nu(dz)dt.$$

Now, we define a stochastic integral with respect to the compensated Poisson random measure $\widetilde{N}$ for the class of predictable and square integrable processes defined below.

Definition 1.2.20. *A stochastic process* $g = \left(g(t,z), z \in \mathbb{R}_0^d\right)_{t\geq 0}$ *is called adapted if* $g(t,z)$ *is* $\mathcal{F}_t$*-measurable for all* $t \geq 0$ *and* $z \in \mathbb{R}_0^d$*. Moreover,* g *is called predictable if it is measurable with respect to the* σ*-field generated by the sets*

$$\left\{B \times (s,t] \times A, B \in \mathcal{F}_s, 0 \leq s < t, A \in \mathscr{B}\left(\mathbb{R}_0^d\right)\right\}.$$

Notice that any adapted and left-continuous process (in t) is predictable.

We denote by $L^2(\mathcal{P})$ the set of stochastic processes $g : \Omega \times \mathbb{R}_+ \times \mathbb{R}_0^d \to \mathbb{R}$ that are predictable and satisfy

$$\int_{\mathbb{R}_+}\int_{\mathbb{R}_0^d} \mathbb{E}\left[g^2(t,z)\right]\nu(dz)dt < \infty.$$

Let $\mathcal{E}$ denote the set of elementary and predictable processes of the form

$$g(t,z) = \sum_{j=1}^{m}\sum_{i=0}^{n-1} F_{i,j}\mathbf{1}_{(t_i,t_{i+1}]}(t)\mathbf{1}_{A_j}(z), \tag{1.18}$$

where $0 \leq t_0 < \cdots < t_n$, all $F_{i,j}$ belong to $\mathcal{F}_{t_i}$ and are bounded, and $A_1, \ldots, A_m$ are pairwise disjoint subsets of $\mathcal{A}_v$. We define the integral of $g \in \mathcal{E}$ of the form (1.18) with respect to $\widetilde{N}$ by

$$\int_{\mathbb{R}_+}\int_{\mathbb{R}_0^d} g(t,z)\widetilde{N}(dt,dz) = \sum_{j=1}^{m}\sum_{i=0}^{n-1} F_{i,j}\widetilde{N}\left((t_i,t_{i+1}], A_j\right). \tag{1.19}$$

This defines a linear functional with the following properties:

Lemma 1.2.21 ([73], Lemma 9.4.1)**.** *For any* $g \in \mathcal{E}$*,*

$$\mathbb{E}\left[\int_{\mathbb{R}_+}\int_{\mathbb{R}_0^d} g(t,z)\widetilde{N}(dt,dz)\right] = 0, \tag{1.20}$$

and

$$\mathbb{E}\left[\left(\int_{\mathbb{R}_+}\int_{\mathbb{R}_0^d} g(t,z)\widetilde{N}(dt,dz)\right)^2\right] = \int_{\mathbb{R}_+}\int_{\mathbb{R}_0^d} \mathbb{E}\left[g^2(t,z)\right]\nu(dz)dt. \tag{1.21}$$

We denote by $L^2_\infty(\mathcal{P})$ the set of stochastic processes $g : \Omega \times \mathbb{R}_+ \times \mathbb{R}_0^d \to \mathbb{R}$ that are predictable and satisfy

$$\int_0^t\int_{\mathbb{R}_0^d} \mathbb{E}\left[g^2(s,z)\right]\nu(dz)ds < \infty,$$

for all $t > 0$. For any process $g \in L^2_\infty(\mathcal{P})$, by considering the restriction of g to each interval $[0;T]$, we can define the indefinite integral process

$$\left(\int_0^t\int_{\mathbb{R}_0^d} g(s,z)\widetilde{N}(ds,dz)\right)_{t\geq 0}.$$

This process is an $\mathcal{F}_t$-martingale (see [2], Theorem 4.2.3). Thus, it is now easy to see that the following properties of stochastic integrals based on Lévy processes:

$$(i) \quad \mathbb{E}\left[\int_0^t \int_{\mathbb{R}_0^d} g(s,z)\widetilde{N}(ds,dz)\right] = 0.$$

$$(ii) \quad \mathbb{E}\left[\int_0^t \int_{\mathbb{R}_0^d} g(s,z)N(ds,dz)\right] = \mathbb{E}\left[\int_0^t \int_{\mathbb{R}_0^d} g(s,z)\nu(dz)ds\right].$$

$$(iii) \quad \mathbb{E}\left[\left(\int_0^t \int_{\mathbb{R}_0^d} g(s,z)\widetilde{N}(ds,dz)\right)^2\right] = \mathbb{E}\left[\int_0^t \int_{\mathbb{R}_0^d} g^2(s,z)\nu(dz)ds\right].$$

We extend the stochastic integral to the set $L^2_{loc}(\mathcal{P})$ of stochastic processes $g : \Omega \times \mathbb{R}_+ \times \mathbb{R}_0^d \to \mathbb{R}$ that are predictable and satisfy

$$\mathbb{P}\left(\int_0^t \int_{\mathbb{R}_0^d} g^2(s,z)\nu(dz)ds < \infty\right) = 1,$$

for all $t > 0$.

Let $g \in L^2_{loc}(\mathcal{P})$. Then we can find a sequence of processes $g^{(n)}$ in $\mathcal{E}$ such that, for all $T > 0$, the sequence

$$\int_0^T \int_{\mathbb{R}_0^d} \left|g(t,z) - g^{(n)}(t,z)\right|^2 \nu(dz)dt$$

converges to zero in probability. Therefore, the sequence

$$\int_0^T \int_{\mathbb{R}_0^d} g^{(n)}(t,z)\widetilde{N}(dt,dz)$$

is Cauchy in probability, and thus converges in probability. We denote the limit by

$$\int_0^T \int_{\mathbb{R}_0^d} g(t,z)\widetilde{N}(dt,dz)$$

and call it the (extended) stochastic integral. The process $(M_t)_{t\geq 0}$ defined by $M_t = \int_0^t \int_{\mathbb{R}_0^d} g(s,z)\widetilde{N}(ds,dz)$ is a local martingale (see [2], Theorem 4.2.12). Therefore, we have the following result.

Corollary 1.2.22 ([73], Corollary 9.4.3)**.** *Let $g \in L^2_{loc}(\mathcal{P})$. Then*

$$\lim_{n\to\infty} \int_0^T \int_{\{|z|\geq 1/n\}} g(s,z)\widetilde{N}(ds,dz) = \int_0^T \int_{\mathbb{R}_0^d} g(s,z)\widetilde{N}(ds,dz),$$

in probability.

Next, we present two important representations of the Lévy process.

Proposition 1.2.23 ([11], Proposition 3.7). **(Lévy-Itô decomposition)** *Let $Z = (Z_t)_{t\geq 0}$ be a Lévy process on $\mathbb{R}^d$ and ν be its Lévy measure.*

- *ν is a Radon measure on $\mathbb{R}_0^d$ and verifies:*
$$\int_{\mathbb{R}_0^d} (1 \wedge |z|^2)\nu(dz) < \infty.$$
- *The jump measure of Z, denoted by N, is a Poisson random measure on $[0,\infty) \times \mathbb{R}^d$ with intensity measure $\nu(dz)dt$.*
- *There exist a vector γ and a d-dimensional Brownian motion $(W_t)_{t\geq 0}$ with covariance matrix A such that*
$$Z_t = \gamma t + W_t + \int_0^t \int_{|z|\geq 1} zN(ds,dz) + \int_0^t \int_{|z|<1} z\widetilde{N}(ds,dz). \tag{1.22}$$

The terms in (1.22) are independent.

Note that, in the case that $Z = (Z_t)_{t\geq 0}$ is a d-dimensional centered pure jump Lévy process whose Lévy measure ν satisfies $\int_{\mathbb{R}^d} \left(1 \wedge |z|^2\right) \nu(dz) < +\infty$, the Lévy-Itô decomposition of Z is given by
$$Z_t = \int_0^t \int_{\mathbb{R}_0^d} z(N(ds,dz) - \nu(dz)ds).$$

Proposition 1.2.24 ([11], Proposition 3.2, Theorem 3.1). **(Lévy-Khinchin representation)** *Let $Z = (Z_t)_{t\geq 0}$ be a Lévy process on $\mathbb{R}^d$. There exists a continuous function $\psi : \mathbb{R}^d \mapsto \mathbb{R}$ called the characteristic exponent of Z, such that:*
$$\mathbb{E}\left[e^{iuZ_t}\right] = e^{t\psi(u)}, \quad u \in \mathbb{R}^d. \tag{1.23}$$
where
$$\psi(u) = i\gamma u - \frac{1}{2}u \cdot Au + \int_{\mathbb{R}^d} \left(e^{iuz} - 1 - iuz\mathbf{1}_{\{|z|\leq 1\}}\right) \nu(dz)$$
The triplet (A,ν,γ) is called *characteristic triplet* or *Lévy triplet* of the process Z_t.

Finally, we now state the Burkholder-Davis-Gundy inequality for the compensated Poisson stochastic integral which will be useful in the thesis.

Lemma 1.2.25 ([2], Theorem 4.4.23 and [89], Proposition 2.2). *Let $\mathscr{B}(\mathbb{R}_0^d)$ be the Borel σ-algebra of $\mathbb{R}_0^d$ and $\mathcal{P}$ be the progressive σ-algebra on $\mathbb{R}_+ \times \Omega$. Let g be a*

$\mathcal{P} \otimes \mathscr{B}(\mathbb{R}_0^d)$-measurable function satisfying that $\int_0^T \int_{\mathbb{R}_0^d} |g(s,z)|^2 \nu(dz)ds < \infty$ $\mathbb{P}-a.s.$ for all $T \geq 0$. Then for any $p \geq 2$, there exists a positive constant C_p such that

$$\mathbb{E}\left[\sup_{t\in[0,T]} \left| \int_0^t \int_{\mathbb{R}_0^d} g(s,z)\widetilde{N}(ds,dz) \right|^p\right]$$
$$\leq C_p \left(\mathbb{E}\left[\left(\int_0^T \int_{\mathbb{R}_0^d} |g(s,z)|^2 \nu(dz)ds \right)^{\frac{p}{2}} \right] + \mathbb{E}\left[\int_0^T \int_{\mathbb{R}_0^d} |g(s,z)|^p \nu(dz)ds \right] \right).$$

Furthermore, for any $1 \leq p < 2$, there exists a positive constant C_p such that

$$\mathbb{E}\left[\sup_{t\in[0,T]} \left| \int_0^t \int_{\mathbb{R}_0^d} g(s,z)\widetilde{N}(ds,dz) \right|^p\right] \leq C_p \mathbb{E}\left[\left(\int_0^T \int_{\mathbb{R}_0^d} |g(s,z)|^2 \nu(dz)ds \right)^{\frac{p}{2}} \right].$$

1.2.5 Stochastic calculus for Lévy processes

Consider processes $X = (X_t)_{t\geq 0}$, admitting stochastic integral representation in the form

$$X_t = x + \int_0^t b(X_s)ds + \int_0^t \sigma(X_s)dW_s + \int_0^t \int_{\mathbb{R}_0} c(X_{s^-}, z)\widetilde{N}(ds,dz). \tag{1.24}$$

Theorem 1.2.26 ([14], Theorem 9.4)**. The one-dimensional Itô formula.** *Let $X = (X_t)_{t\geq 0}$, be the Itô-Lévy process given by (1.24) and let $f : (0,\infty) \times \mathbb{R} \to \mathbb{R}$ be a function in $C^{1,2}((0,\infty)\times\mathbb{R})$ and define*

$$Y_t := f(t, X_t), \quad t \geq 0.$$

Then the process $Y = (Y_t)_{t\geq 0}$, is also an Itô-Lévy process and its differential form is given by

$$\begin{aligned} dY_t = &\frac{\partial f}{\partial t}(t,X_t)dt + \frac{\partial f}{\partial x}(t,X_t)b(X_t)dt + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(t,X_t)\sigma^2(X_t)dt \\ &+ \frac{\partial f}{\partial x}(t,X_t)\sigma(X_t)dW_t \\ &+ \int_{\mathbb{R}_0} \left[f(t, X_t + c(X_t,z)) - f(t,X_t) - \frac{\partial f}{\partial x}(t,X_t)c(X_t,z) \right] \nu(dz)dt \\ &+ \int_{\mathbb{R}_0} \left[f\left(t, X_{t^-} + c(X_{t^-},z)\right) - f\left(t, X_{t^-}\right) \right] \widetilde{N}(dt,dz). \end{aligned} \tag{1.25}$$

In the multidimensional case, we are given a d-dimensional Brownian motion $W_t = \left(W_t^1, \ldots, W_t^d\right)^T, t \geq 0$, and independent compensated Poisson random measures $\widetilde{N}(dt,dz) =$

$\left(\widetilde{N}_1(dt,dz_1),\ldots,\widetilde{N}_d(dt,dz_d)\right)^T, t\geq 0,\ z=(z_1,\ldots,z_d)\in\ \mathbb{R}_0^d$, and n Itô-Lévy processes of the form

$$dX_t = b(X_t)dt + \sigma(X_t)dW_t + \int_{\mathbb{R}_0^d} c(X_{t^-},z)\widetilde{N}(dt,dz), \quad t\geq 0,$$

that is, for $i=1,\ldots,n$

$$dX_t^i = b_i(X_t)dt + \sum_{j=1}^d \sigma_{ij}(X_t)dW_t^j + \sum_{j=1}^d \int_{\mathbb{R}_0} c_{ij}\left(X_{t^-},z_j\right)\widetilde{N}_j\left(dt,dz_j\right). \tag{1.26}$$

Theorem 1.2.27 ([14], Theorem 9.5)**. The multi-dimensional Itô formula.** *Let $X=(X_t)_{t\geq 0}$, be an d-dimensional Itô-Lévy process of the form (1.26). Let $f:(0,\infty)\times\mathbb{R}^d\to\mathbb{R}$ be a function in $C^{1,2}\left((0,\infty)\times\mathbb{R}^d\right)$ and define*

$$Y_t := f(t,X_t), \quad t\geq 0.$$

Then the process $Y=(Y_t)_{t\geq 0}$, is a one-dimensional Itô-Lévy process and its differential form is given by

$$\begin{aligned}
& dY(t)\\
=& \frac{\partial f}{\partial t}(t,X_t)dt + \sum_{i=1}^n \frac{\partial f}{\partial x_i}(t,X_t)b_i(X_t)dt + \sum_{i=1}^d\sum_{j=1}^d \frac{\partial f}{\partial x_i}(t,X_t)\sigma_{ij}(X_t)dW_t^j\\
& + \frac{1}{2}\sum_{i=1}^n\sum_{j=1}^d \frac{\partial^2 f}{\partial x_i\partial x_j}(t,X_t)\left(\sigma\sigma^T\right)_{ij}(X_t)dt + \sum_{j=1}^d\int_{\mathbb{R}_0}\Bigg[f\left(t,X_t+c^{(j)}(X_t,z)\right)\\
& - f(t,X_t) - \sum_{i=1}^d \frac{\partial f}{\partial x_i}(t,X_t)c_{ij}(X_t,z)\Bigg]\nu_j\left(dz_j\right)dt\\
& + \sum_{j=1}^d\int_{\mathbb{R}_0}\left[f\left(t,X_{t^-}+c^{(j)}(X_{t^-},z)\right) - f\left(t,X_{t^-}\right)\right]\widetilde{N}_j\left(dt,dz_j\right),
\end{aligned} \tag{1.27}$$

where $c^{(j)}$ is the column number j of the $d\times d$ matrix $c=[c_{ij}]$.

1.2.6 Stochastic differential equations driven by Lévy processes

We next consider a process $X=(X_t)_{t\geq 0}$, the solution to the following stochastic differential equation with jumps in $\mathbb{R}^d$:

$$dX_t = b\left(X_t\right)dt + \sum_{j=1}^d \sigma_j\left(X_t\right)dW_t^j + \int_{\mathbb{R}_0} c\left(X_{t-},z\right)\widetilde{N}(dt,dz),$$

with initial condition $X_0 = x_0 \in \mathbb{R}^d$. The coefficients $\sigma_j, b : \mathbb{R}^d \to \mathbb{R}^d$ and $c : \mathbb{R}^d \times \mathbb{R}_0 \to \mathbb{R}^d$ are measurable functions satisfying the following Lipschitz and linear growth conditions: for all $x, y \in \mathbb{R}^d$,

$$\max_j \left\{ |\sigma_j(x) - \sigma_j(y)|^2 ; |b(x) - b(y)|^2; \int_{\mathbb{R}_0} |c(x,z) - c(y,z)|^2 \nu(dz) \right\} \le K|x-y|^2$$

and

$$\max_j \left\{ |\sigma_j(x)|^2 ; |b(x)|^2; \int_{\mathbb{R}_0} |c(x,z)|^2 \nu(dz) \right\} \le K \left(1 + |x|^2\right).$$

Theorem 1.2.28 ([73], Theorem 11.4.2)**.** *There exists a unique càdlàg, adapted, and Markov process X on $(\Omega, \mathcal{F}, \mathbb{P})$ satisfying the integral equation*

$$X_t = x_0 + \int_0^t b(X_s)\, ds + \sum_{j=1}^d \int_0^t \sigma_j(X_s)\, dW_s^j + \int_0^t \int_{\mathbb{R}_0} c(X_{s-}, z)\, \widetilde{N}(ds, dz).$$

Moreover, for any $T > 0$,

$$\mathbb{E}\left[\sup_{t\in[0;T]} |X_t|^2\right] < \infty.$$

1.3 Numerical analysis for stochastic differential equations with irregular coefficients

Consider the d-dimensional stochastic differential equation of Itô type

$$dX_t = b(t, X_t)dt + \sigma(t, X_t)dW_t \quad \text{on } t_0 \le t \le T \tag{1.28}$$

with initial value $x(t_0) = x_0$.

The foundational approach to discretizing equation (1.28) is the Euler-Maruyama (EM) scheme, first formally introduced by Maruyama in his seminal 1955 paper [64]. This work established the fundamental concept of a step-by-step approximation for SDEs, defined as follows:

Theorem 1.3.1 ([64], Theorem 1)**.** *Let $W(t)$ be the Brownian motion, $W(0) = 0$, $\mathbb{E}\left[(\Delta W_t)^2\right] = \Delta t$, of which Gaussian increments ΔW_t over disjoint intervals are independent, and let $b(t,x)$, $\sigma(t,x)$ be continuous functions satisfying the Lipschitz condition. Consider a division of $(0;1), \Delta = \Delta(t_0, t_1, \ldots, t_n), 0 = t_0 < t_1 < \cdots < t_n = 1$,* *and variables* $X_1, X_2, \ldots, X_n$ *defined by*

$$\begin{cases} X_1 = X_0 + b(t_0, X_0)\,\Delta t_0 + \sigma(t_0, X_0)\,\Delta W_0 \\ X_2 = X_1 + b(t_1, X_1)\,\Delta t_1 + \sigma(t_1, X_1)\,\Delta W_1 \\ \cdots\cdots\cdots\cdots\cdots \\ X_n = X_{n-1} + b(t_{n-1}, X_{n-1})\,\Delta t_{n-1} + \sigma(t_{n-1}, X_{n-1})\,\Delta W_{n-1} \end{cases}$$

and also $X_\Delta(t)$ *defined by*

$$X_\Delta(t) = X_\mu + b(t_\mu, X_\mu)(t - t_\mu) + \sigma(t_\mu, X_\mu)\left[W_t - W_{t_\mu}\right]$$

where

$$W_v = W_{t_v}, \quad \Delta W_{v-1} = W_v - W_{v-1}, \quad \Delta t_{v-1} = t_v - t_{v-1} \quad t_\mu \leqslant t < t_{\mu+1}$$

Then, for every t*, in the* L^2*-sense, there exists*

$$X(t) = \lim_{\rho(\Delta)\to 0} X_\Delta(t), \quad \rho(\Delta) = \max_{1\leq v\leq n} \Delta t_{v-1}$$

and $X(t)$ *becomes the unique solution of* (1.28)*.*

While Maruyama established the convergence, the rate of this convergence under standard regularity conditions was rigorously analyzed by Kloeden and Platen [46]. They demonstrated that if the coefficients satisfy a global Lipschitz condition, the EM scheme achieves a strong convergence order of $1/2$. The analog of the Euler-Maruyama approximation for (1.28) is defined by

$$X_0^h = x, \quad X_{t_{k+1}}^h = X_{t_k}^h + b\left(t_k, X_{t_k}^h\right)h + \sigma\left(t_k, X_{t_k}^h\right)\Delta W_k$$
$$\Delta W_k = \Delta W_k^h = W_{t_{k+1}} - W_{t_k}, \quad t_k = kh$$

It is convenient to extend it to the whole interval $[0; T]$ as follows:

$$\begin{cases} X_0^h = x \\ X_t^h = X_{t_k}^h + b\left(t_k, X_{t_k}^h\right)(t - t_k) + \sigma\left(t_k, X_{t_k}^h\right)(W_t - W_{t_k}), \quad t \in [t_k, t_{k+1}] \end{cases}$$

Theorem 1.3.2 ([46], Theorem 13.2)**.** *Suppose that the coefficients of equation* (1.28) *satisfy the Lipschitz condition*

$$|b(t,x) - b(s,y)|^2 + |\sigma(t,x) - \sigma(s,y)|^2 \leqslant C\left(|x-y|^2 + |t-s|\right)$$

Then

$$\sup_{t\leqslant T} \mathbb{E}\left[\left|X_t^h - X_t\right|\right] = O\left(h^{1/2}\right), \quad h \to 0$$

that is, the Euler-Maruyama approximation (as a strong approximation) has the order $\frac{1}{2}$.

The global Lipschitz condition, however, is often too restrictive for many practical models. A significant body of research addresses SDEs with irregular coefficients. Gyöngy and Rásonyi [25] tackled the case where the diffusion coefficient is not Lipschitz but merely Hölder continuous. They showed that the standard EM scheme still converges, albeit at a reduced rate dependent on the Hölder exponent. In [25], the authors construct the Euler-Maruyama approximation scheme as follows:

For integers $n \geq 1$, let us define the functions $\kappa_n : [0;T] \to [0;T]$ by $\kappa_n(T) := \frac{n-1}{n}T$ and

$$\kappa_n(x) = \frac{iT}{n} \quad \text{for } \frac{iT}{n} \leq x < \frac{(i+1)T}{n}, \text{ for } i = 0, \dots, n-1.$$

Let define the Euler approximations of $X(t), t \in [0;T]$, as the solutions of

$$\begin{aligned} dX_n(t) &= b\left(t, X_n\left(\kappa_n(t)\right)\right) dt + \sigma\left(t, X_n\left(\kappa_n(t)\right)\right) dW(t) \\ X_n(0) &= \xi \end{aligned}$$

for each $n \geq 1$.

Theorem 1.3.3 ([25], Theorem 2.1)**.** *Assume that $\sigma, f, g : [0;T] \times \mathbb{R} \to \mathbb{R}$ are measurable; $g(t, \cdot)$ is monotone decreasing; $b = f + g$ and there exist $K > 0, \alpha \in [0, 1/2]$ and $\gamma \in (0, 1]$ such that for all $t \in [0;T]$ and $x, y \in \mathbb{R}$,*

$$\begin{aligned} &|\sigma(t,x) - \sigma(t,y)| \leq K|x-y|^{\frac{1}{2}+\alpha}, \quad |f(t,x) - f(t,y)| \leq K|x-y| \\ &|g(t,x) - g(t,y)| \leq K|x-y|^{\gamma}, \end{aligned}$$

and

$$|b(t,0)| + |\sigma(t,0)| \leq K.$$

Let $\mathbb{E}\left[|\xi|^{1+2\alpha}\right] < \infty$. Then there is a constant C depending only on K, T, γ and $\mathbb{E}\left[|\xi|^{1+2\alpha}\right]$ such that

$$\mathbb{E}\left[|X(\tau) - X_n(\tau)|\right] \leq \begin{cases} \frac{C}{\ln n} & \text{if } \alpha = 0, \\ C\left(\frac{1}{n^{\alpha}} + \frac{1}{n^{\gamma/2}}\right) & \text{if } \alpha \in (0, 1/2]. \end{cases}$$

for all $n \geq 2$ for every stopping time $\tau \leq T$.

A more severe irregularity occurs when the drift coefficient exhibits super-linear growth. In this scenario, the standard EM scheme is known to diverge in the L^p-sense. To overcome this divergence, a new class of explicit tamed schemes was developed.

Hutzenthaler, Jentzen, and Kloeden [35] proposed a seminal "tamed" Euler-Maruyama scheme where the drift coefficient is bounded in the numerical update step. In [35], the tamed Euler-Maruyama approximation scheme is constructed as follows. Let $Y_n^N : \Omega \to \mathbb{R}^d, n \in \{0, 1, \ldots, N\}, N \in \mathbb{N}$, be given by $Y_0^N = \xi$ and

$$Y_{n+1}^N = Y_n^N + \frac{T/N \cdot b\left(Y_n^N\right)}{1 + T/N \cdot |b\left(Y_n^N\right)|} + \sigma\left(Y_n^N\right)\left(W_{(n+1)T/N} - W_{nT/N}\right) \tag{1.29}$$

for all $n \in \{0, 1, \ldots, N-1\}$ and all $N \in \mathbb{N}$.

In order to formulate the convergence theorem for the tamed Euler method (1.29), we now introduce appropriate time−continuous interpolations of the time discrete numerical approximations (1.29). More formally, let $\bar{Y}^N : [0;T] \times \Omega \to \mathbb{R}^d, N \in \mathbb{N}$, be a sequence of stochastic processes given by

$$\bar{Y}_t^N = Y_n^N + \frac{(t - nT/N) \cdot b\left(Y_n^N\right)}{1 + T/N \cdot |b\left(Y_n^N\right)|} + \sigma\left(Y_n^N\right)\left(W_t - W_{nT/N}\right) \tag{1.30}$$

for all $t \in \left[\frac{nT}{N}, \frac{(n+1)T}{N}\right], n \in \{0, 1, \ldots, N-1\}$ and all $N \in \mathbb{N}$.

Theorem 1.3.4 ([35], Theorem 1.1)**.** *Let $b : \mathbb{R}^d \to \mathbb{R}^d$ be a continuously differentiable and globally one-sided Lipschitz continuous function whose derivative grows at most polynomially and let $\sigma = (\sigma_{ij})_{i\in\{1,2,\ldots,d\}, j\in\{1,2,\ldots,m\}} : \mathbb{R}^d \to \mathbb{R}^{d\times m}$ be a globally Lipschitz continuous function. More formally, suppose that there is a real number $c \in (0, \infty)$ such that $|b'(x)| \le c(1 + |x|^c), |\sigma(x) - \sigma(y)| \le c|x-y|$ and $\langle x-y, b(x)-b(y)\rangle \le c|x-y|^2$ for all $x, y \in \mathbb{R}^d$. Then there exists a family $C_p \in [0, \infty), p \in [1, \infty)$, of real numbers such that*

$$\left(\mathbb{E}\left[\sup_{t\in[0;T]} \left|X_t - \bar{Y}_t^N\right|^p\right]\right)^{1/p} \le C_p N^{-1/2},$$

for all $N \in \mathbb{N}$ and all $p \in [1, \infty)$. Here $X : [0;T] \times \Omega \to R^d$ is the exact solution of the SDE (1.28) and $\bar{Y}^N : [0;T] \times \Omega \to R^d, N \in N$, are the time continuous interpolations (1.30) of the numerical approximations (1.29).

Concurrently, Sabanis [79] proposed a similar tamed scheme, also designed to control super-linear drift, where the taming function explicitly depends on a step-size parameter α. For every $n \ge 1$, and any $t \in [0;T]$, the following tamed Euler scheme is defined

$$dX_n(t) = b_n\left(t, X_n\left(\kappa_n(t)\right)\right) dt + \sigma\left(t, X_n\left(\kappa_n(t)\right)\right) dW(t)$$

with the same initial value $X(0)$ as SDE (1.28) and $\kappa_n(t) := [nt]/n$. Moreover, it is assumed that

$$b_n(t,x) := \frac{1}{1+n^{-\alpha}|b(t,x)|}b(t,x),$$

for any $t \in [0;T], x \in \mathbb{R}^d$ and $\alpha \in (0,1/2]$.

Theorem 1.3.5 ([79], Corollary 2.3). *Suppose the drift coefficient is superlinearly growing and the diffusion coefficient is locally Lipschitz continuous and has at most linear growth, then the tamed Euler scheme with $\alpha = 1/2$ converges to the true solution of SDE (1.28) in L^p-sense with order $1/2$, i.e.*

$$\mathbb{E}\left[\sup_{0\le t\le T} |X(t) - X_n(t)|^p\right] \le Cn^{-p/2}$$

for all $p > 0$, where C is a constant independent of n.

Building on these developments, our research group [43] addressed the challenging scenario where both types of irregularities are present: a one-sided Lipschitz drift (allowing for super-linear growth) and a Hölder continuous diffusion coefficient. We proposed a novel tamed scheme that modifies both the drift and diffusion coefficients to ensure convergence.

For each $h \in \left(0, \frac{L_1}{L_2^2}\right)$, denote $\eta_h(t) = kh$ if $t \in [kh,(k+1)h)$ for some $k = 0, 1, \ldots$, and

$$b_h(x) = \frac{b(x)}{1 - L_2^2 L_1^{-1} h} \quad \text{and} \quad \sigma_h(t,x) = \frac{\sigma(x)}{1 + h^{1/2} e^{2L_1 t}(|\sigma(x)| + 1)}$$

A tamed Euler-Maruyama approximation of SDE (1.28) is defined as follows

$$X_t^h = x_0 + \int_0^t b_h\left(X_{\eta_h(s)}^h\right) ds + \int_0^t \sigma_h\left(\eta_h(s), X_{\eta_h(s)}^h\right) dW_s, \quad t \in [0,+\infty) \tag{1.31}$$

Theorem 1.3.6 ([43], Theorem 1). *(i) Assume that b satisfies the local Lipschitz condition; b and σ satisfy the linear growth condition. For any $T > 0$,*

$$\lim_{h\to 0} \mathbb{E}\left[\sup_{0\le t\le T} \left|X_t^h - X_t\right|\right] = 0.$$

(ii) Assume that b satisfies the local Lipschitz condition and σ satisfies the Hölder condition. If $0 < h < \frac{L_1}{2L_2^2} \wedge 1$ then there exists a positive constant $C = C(x_0, L_2, L_3, T, \alpha)$ such that

$$\sup_{0\le t\le T} \mathbb{E}\left[\left|X_t^h - X_t\right|\right] \le \begin{cases} Ch^\alpha & \text{if } 0 < \alpha \le \frac{1}{2} \\ \frac{C}{\log(1/h)} & \text{if } \alpha = 0 \end{cases},$$

and

$$\mathbb{E}\left[\sup_{0\le t\le T}\left|X_t^h - X_t\right|\right] \le \begin{cases} Ch^{2\alpha^2} & \text{if } 0<\alpha\le\frac{1}{2} \\ \frac{C}{\sqrt{\log(1/h)}} & \text{if } \alpha = 0 \end{cases}.$$

(iii) Assume that b satisfies the local Lipschitz condition and σ satisfies the Hölder condition. If $0 < h < \frac{L_1}{2L_2^2} \wedge 1$, for any $p \ge 2$, there exists a constant $C = C(x_0, L_2, L_3, T, p, \alpha)$ such that

$$\mathbb{E}\left[\sup_{0\le t\le T}\left|X_t - X_t^h\right|^p\right] \le \begin{cases} \frac{C}{\log(1/h)} & \text{if } \alpha = 0 \\ Ch^{p/2} & \text{if } \alpha = \frac{1}{2} \\ Ch^{\alpha} & \text{if } 0<\alpha<\frac{1}{2} \end{cases}.$$

The tamed Euler-Maruyama scheme (1.31) converges in L^1-norm, L^1-sup norm, and L^p-sup norm at the same rates as the plain Euler-Maruyama scheme does when applying for SDEs with Hölder continuous coefficients (see [25]).

A critical limitation of all the aforementioned works is their focus on convergence over a finite time horizon [0,T]. Consequently, approximation over infinite time intervals, which is essential for analyzing long-term behavior and stability, has garnered significant recent interest. A pivotal contribution in this direction is the adaptive Euler-Maruyama scheme introduced by Fang and Giles [18]. The core idea is to replace the fixed step size h with an adaptive step size $h_n \triangleq h\left(\hat{X}_{t_n}\right)$, which contracts based on the current state of the process. The proposed adaptive EulerMaruyama discretisation is

$$t_{n+1} = t_n + h_n, \quad \widehat{X}_{t_{n+1}} = \widehat{X}_{t_n} + b(\widehat{X}_{t_n})h_n + \sigma(\widehat{X}_{t_n})\Delta W_n$$

where $h_n \triangleq h(\widehat{X}_{t_n})$ and $\Delta W_n \triangleq W_{t_{n+1}} - W_{t_n}$, and there is fixed initial data $t_0 = 0, \widehat{X}_0 = X_0$.

We use the notation $\underline{t} \triangleq \max\{t_n : t_n \le t\}, n_t \triangleq \max\{n : t_n \le t\}$ for the nearest time point before time t, and its index. We define the piecewise constant interpolant process $\overline{X}_t = \widehat{X}_{\underline{t}}$ and also define the standard continuous interpolant as

$$\widehat{X}_t = \widehat{X}_{\underline{t}} + b(\widehat{X}_{\underline{t}})(t-\underline{t}) + \sigma(\widehat{X}_{\underline{t}})\left(W_t - W_{\underline{t}}\right)$$

so that $\widehat{X}_t$ is the solution of the SDE

$$d\widehat{X}_t = b(\widehat{X}_{\underline{t}})dt + \sigma(\widehat{X}_{\underline{t}})dW_t = b(\overline{X}_t)dt + \sigma(\overline{X}_t)dW_t.$$

Theorem 1.3.7 ([18], Theorem 3)**.** *If b satisfies the one-sided Lipschitz condition and the polynomial growth Lipschitz condition; σ satisfies the Lipschitz condition and the timestep functions h, h^δ satisfy*

$$\langle x, b(x)\rangle + \frac{1}{2}h(x)|b(x)|^2 \leq \alpha|x|^2 + \beta,$$
$$\delta \min(T, h(x)) \leq h^\delta(x) \leq \min(\delta T, h(x)).$$

Then for all $p > 0$, there exists a constant $C_{p,T}$ such that

$$\mathbb{E}\left[\sup_{0\leq t\leq T}\left|\widehat{X}_t - X_t\right|^p\right] \leq C_{p,T}\delta^{p/2}.$$

Theorem 1.3.8 ([18], Theorem 6)**.** *Assume that b and σ satisfy the contractive Lipschitz condition, b satisfies the polynomial growth Lipschitz condition and σ satisfies the Lipschitz condition. Moreover, assume that the timestep functions h, h^δ satisfies*

$$\langle x, b(x)\rangle + \frac{1}{2}h(x)|b(x)|^2 \leq -\alpha|x|^2 + \beta,$$
$$\delta \min(T, h(x)) \leq h^\delta(x) \leq \min(\delta T, h(x)).$$

Then for all $p \in (0, p^*]$ there exists a constant C_p such that, $\forall t \geq 0$,

$$\mathbb{E}\left[\left|\widehat{X}_t - X_t\right|^p\right] \leq C_p\delta^{p/2}.$$

Chapter 2

TAMED-ADAPTIVE EULER-MARUYAMA SCHEME FOR LÉVY-DRIVEN SDEs WITH IRREGULAR COEFFICIENTS

This chapter introduces a novel approximation scheme for a class of stochastic differential equations (SDEs) with irregular coefficients. We address both SDEs driven by Brownian motion (continuous noise) and the more general case of SDEs driven by Lévy processes (which include jumps). Since the Brownian-driven model is a special case of the Lévy model and the proof methods are logically parallel, this chapter, for the sake of conciseness, presents the detailed analysis only for the more general Lévy case. The corresponding results for the simpler Brownian-driven model can be inferred similarly. The results presented herein are based on the author's publications [1, 2] listed in the **List of Author's Related Papers** section.

2.1 Introduction

We consider the process $X = (X_t)_{t\geq 0}$ as a solution to the following stochastic differential equation (SDE) with jumps

$$X_t = x_0 + \int_0^t b(X_s)ds + \int_0^t \sigma(X_s)dW_s + \int_0^t c\left(X_{s-}\right)dZ_s, \tag{2.1}$$

for $t \geq 0$, where $x_0 \in \mathbb{R}$, $W = (W_t)_{t\geq 0}$ is a one-dimensional standard Brownian motion and $Z = (Z_t)_{t\geq 0}$ is a one-dimensional centered pure jump Lévy process (independent of W) with Lévy measure ν satisfying $\int_{\mathbb{R}}(1 \wedge z^2)\nu(dz) < +\infty$. Two processes W and Z are defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ equipped with the natural filtration $(\mathcal{F}_t)_{t\geq 0}$ generated by W and Z and augmented by all the null sets in $\mathcal{F}$ so that it satisfies the usual conditions. The integral equation of (2.1) can be written as

$$X_t = x_0 + \int_0^t b(X_s)ds + \int_0^t \sigma(X_s)dW_s + \int_0^t \int_{\mathbb{R}_0} c\left(X_{s-}\right) z\widetilde{N}(ds, dz).$$

Such Lévy-driven SDEs arise in many applications [11, 75] and the references therein. Therefore, it is important to find effective methods to solve such SDEs numerically. Several modified Euler-Maruyama schemes have been proposed for Lévy-driven SDEs with locally Lipschitz and super-linearly growing coefficients in [9, 10, 12, 13, 29, 30, 31, 52, 53, 56]. The first explicit approximation for SDEs driven by Brownian motion with super-linearly growing drift coefficients is the tamed Euler-Maruyama scheme, which was introduced in [35, 36, 37, 79]. The strong convergence of Euler-Maruyama schemes for Lévy-driven SDEs with Hölder continuous diffusion coefficient has been studied in [54, 55, 88].

Crucially, all the works mentioned above only considered the convergence of the approximation scheme in a finite time interval, say $[0; T]$ with $T < \infty$. For SDEs driven only by Brownian motions, the approximation in an infinite time interval has only recently been studied by Fang and Giles [18]. They introduced an adaptive Euler-Maruyama approximation scheme and showed its strong convergence in the interval $[0, \infty)$ when applied for SDEs whose coefficients b and σ satisfy the contractive Lipschitz condition, b is locally Lipschitz and of polynomial growth, and σ is globally Lipschitz continuous. In this chapter, we propose a tamed-adaptive Euler-Maruyama approximation scheme for Lévy-driven stochastic differential equations with locally Lipschitz continuous, polynomial growth drift, and locally Hölder continuous, polynomial growth diffusion coefficients. Moreover, we show that the new scheme converges in both finite and infinite time intervals under some suitable conditions on the regularity and the growth of the coefficients.

2.2 Model assumptions

In this chapter, we assume that the drift, diffusion and jump coefficients b, σ, c and the Lévy measure ν of equation (2.1) satisfy the following conditions:

B1. There exists a positive constant L_0 such that

$$|c(x)| \leq L_0(1 + |x|),$$

for any $x \in \mathbb{R}$.

B2. For some $p_0 \in [2; +\infty)$, there exist constants $\gamma \in \mathbb{R}$, $\eta \in [0; +\infty)$ such that

$$xb(x) + \frac{p_0 - 1}{2}\sigma^2(x) + \frac{c^2(x)}{2L_0}\int_{\mathbb{R}_0} |z| \left((1 + L_0|z|)^{p_0-1} - 1\right) \nu(dz) \leq \gamma x^2 + \eta,$$

for any $x \in \mathbb{R}$.

B3. Coefficient b is locally Lipschitz: for any $R > 0$, there exists a positive constant L_R such that

$$|b(x) - b(y)| \leq L_R|x - y|,$$

for all $|x| \vee |y| \leq R$.

B4. Coefficient σ is locally $\left(\alpha + \frac{1}{2}\right)$ −Hölder continuous: for any $R > 0$, there exist positive constants L_R and $\alpha \in \left(0, \frac{1}{2}\right]$ such that

$$|\sigma(x) - \sigma(y)| \leq L_R|x - y|^{1/2+\alpha},$$

for all $|x| \vee |y| \leq R$.

B5. Coefficient c is locally Lipschitz: for any $R > 0$, there exists a positive constant L_R such that

$$|c(x) - c(y)| \leq L_R|x - y|,$$

for all $|x| \vee |y| \leq R$.

B6. $\int_{|z|>1} |z|^p \nu(dz) < \infty$ for all $p \in [1; 2p_0]$ and $\int_{0<|z|\leq 1} |z|\nu(dz) < \infty$.

B7. Coefficient b is one-sided Lipschitz: there exists a constant L_1 such that

$$(x - y)(b(x) - b(y)) \leq L_1|x - y|^2,$$

for any $x, y \in \mathbb{R}$.

B8. Coefficient b is locally Lipschitz continuous: there exist positive constants l and L_2 such that

$$|b(x) - b(y)| \leq L_2 \left(1 + |x|^l + |y|^l\right) |x - y|,$$

for any $x, y \in \mathbb{R}$.

B9. Coefficient σ is $(\alpha + \frac{1}{2})$−locally Hölder continuous: there exist positive constants m, L_3 and $\alpha \in [0; \frac{1}{2}]$ such that

$$|\sigma(x) - \sigma(y)| \leq L_3 (1 + |x|^m + |y|^m) |x - y|^{1/2+\alpha},$$

for any $x, y \in \mathbb{R}$.

B10. Coefficient c is Lipschitz: there exists a positive constant L_4 such that

$$|c(x) - c(y)| \leq L_4|x - y|,$$

for any $x, y \in \mathbb{R}$.

Remark 2.2.1. It is crucial to clarify the roles of these assumptions, particularly their relationship to the foundational Brownian-driven case presented in our publication [1] listed in the **List of Author's Related Papers** section.

(i) **Conditions for existence and numerical analysis:** We distinguish two levels of regularity. Assumptions **B3**, **B4**, and **B5** state standard local Lipschitz and local Hölder continuity. As noted in the dissertation (and based on [69]), these weaker conditions are sufficient to guarantee the existence and uniqueness of the strong solution. In contrast, Assumptions **B7, B8, B9,** and **B10** provide stronger, quantitative bounds (one-sided Lipschitz and polynomial-growth). These explicit bounds are not required for existence but are essential for the numerical analysis — specifically, to control the moments and the convergence error of the approximation scheme.

(ii) **Generalization from the Brownian-driven case:** This framework is a direct generalization of the author's prior work on SDEs without jumps.

- **Inherited conditions:** The core assumptions for the drift and diffusion, **B7** (one-sided Lipschitz), **B8** (poly-growth local Lipschitz), and **B9** (poly-growth local Hölder), are inherited directly from Assumptions A2, A3, and A4 in our first publication.
- **Novelty for Lévy processes:** The primary contribution here is the extension to the Lévy-driven setting. This requires:
  - Regularity for the new jump coefficient c (Assumption **B10**).
  - Most importantly, Assumption **B2** is the non-trivial generalization of the key moment-controlling condition **A1** found in our first publication. The original condition **A1** ($xb(x)+\frac{p_0-1}{2}|\sigma(x)|^2 \leq \gamma|x|^2+\eta$) is insufficient in the presence of jumps. **B2** incorporates the necessary compensating integral term (derived from the Itô-Lévy formula for $V(x) = |x|^{p_0}$) to ensure the p_0-th moment remains bounded.

2.3 Lévy-driven SDEs with irregular coefficients

In this section, we first establish the existence and uniqueness of the solution under our specific set of assumptions **B1–B5**. This result extends the existing literature [23, 58, 85] to accommodate the challenging combination of locally Hölder diffusion and super-linearly growing coefficients.

First, we establish a priori estimate for the moments of X_t as long as the solution exists.

Proposition 2.3.1. *Assume that coefficients b, c, σ and the Lévy measure ν satisfy conditions **B1, B2, B6** and σ is bounded on every compact subset of $\mathbb{R}$. Assume further that $X = (X_t)_{t\geq 0}$ is a solution to equation (2.1). Then, for any $p \in (0, p_0]$, there exists a positive constant C_p such that for any $t \geq 0$,*

$$\mathbb{E}\left[|X_t|^p\right] \leq \begin{cases} C_p(1+e^{\gamma p t}) & \text{if } \gamma \neq 0, \\ C_p(1+t)^{p/2} & \text{if } \gamma = 0. \end{cases} \tag{2.2}$$

Note that when $\gamma < 0$, we have $\sup_{t\geq 0} \mathbb{E}\left[|X_t|^p\right] \leq 2C_p$.

Remark 2.3.2. Since $(1 + L_0|z|)^x$ is an increasing function for $x \geq 1$, it follows from Condition **B2** that for any $p \in [2; p_0]$ and $x \in \mathbb{R}$,

$$xb(x) + \frac{p-1}{2}\sigma^2(x) + \frac{c^2(x)}{2L_0}\int_{\mathbb{R}_0} |z| \left((1+L_0|z|)^{p-1} - 1\right)\nu(dz) \leq \gamma x^2 + \eta.$$

Proof of Proposition 2.3.1. Let $p \in [2, p_0]$ be an even natural number. Applying Itô's formula to $e^{-p\gamma t}X_t^p$, we have for any $t \geq 0$,

$$\begin{aligned} e^{-p\gamma t}X_t^p = x_0^p &+ p\int_0^t e^{-p\gamma s}\left(-\gamma X_s^p + X_s^{p-1}b\left(X_s\right) + \frac{p-1}{2}X_s^{p-2}\sigma^2\left(X_s\right)\right)ds \\ &+ \int_0^t\int_{\mathbb{R}_0} e^{-p\gamma s}\left[\left(X_s + c\left(X_s\right)z\right)^p - X_s^p - pX_s^{p-1}c\left(X_s\right)z\right]\nu(dz)ds \\ &+ p\int_0^t e^{-p\gamma s}X_s^{p-1}\sigma\left(X_s\right)dW_s \\ &+ \int_0^t\int_{\mathbb{R}_0} e^{-p\gamma s}\left[\left(X_{s-} + c\left(X_{s-}\right)z\right)^p - X_{s-}^p\right]\widetilde{N}(ds,dz). \end{aligned} \tag{2.3}$$

Then, applying (2.3) to $p = 2$, using **B2** and Remark 2.3.2, we get

$$e^{-2\gamma t}X_t^2$$

$$
\begin{aligned}
&= x_0^2 + 2\int_0^t e^{-2\gamma s}\left[-\gamma X_s^2 + X_s b\left(X_s\right) + \frac{1}{2}\sigma^2\left(X_s\right) + \frac{1}{2}c^2\left(X_s\right)\int_{\mathbb{R}_0} z^2\nu\left(dz\right)\right]ds \\
&\quad + 2\int_0^t e^{-2\gamma s}X_s\sigma\left(X_s\right)dW_s \\
&\quad + \int_0^t\int_{\mathbb{R}_0} e^{-2\gamma s}\left[2X_{s-}c\left(X_{s-}\right)z + c^2\left(X_{s-}\right)z^2\right]\widetilde{N}(ds,dz) \\
&\le x_0^2 + 2\eta\int_0^t e^{-2\gamma s}ds + 2\int_0^t e^{-2\gamma s}X_s\sigma\left(X_s\right)dW_s \\
&\quad + \int_0^t\int_{\mathbb{R}_0} e^{-2\gamma s}\left[2X_{s-}c\left(X_{s-}\right)z + c^2\left(X_{s-}\right)z^2\right]\widetilde{N}(ds,dz). \qquad (2.4)
\end{aligned}
$$

Now, for each $N > 0$, we denote $\tau_N := \inf\{t \geq 0 : |X_t| \geq N\}$. Then, using (2.4), the fact that σ is bounded on every compact subset of $\mathbb{R}$ and Condition **B1**, we obtain

$$
\mathbb{E}\left[e^{-2\gamma(t\wedge\tau_N)}X_{t\wedge\tau_N}^2\right] \le x_0^2 + 2\eta\int_0^t e^{-2\gamma s}ds. \qquad (2.5)
$$

This implies that

$$
\mathbb{P}(\tau_N < t) \le \left(x_0^2 + 2\eta\int_0^t e^{-2\gamma s}ds\right)e^{2\gamma t}N^{-2},
$$

which deduces that $\tau_N \uparrow \infty$ a.s. as $N \uparrow \infty$. Now, let $N \uparrow \infty$ and using Fatou's lemma for the left-hand side of (2.5), we obtain

$$
\mathbb{E}\left[e^{-2\gamma t}X_t^2\right] \le x_0^2 + 2\eta\int_0^t e^{-2\gamma s}ds.
$$

<u>If $\gamma = 0$:</u>

$$
\mathbb{E}\left[X_t^2\right] \le x_0^2 + 2\eta t.
$$

<u>If $\gamma \neq 0$:</u>

$$
\mathbb{E}\left[X_t^2\right] \le \left(x_0^2 + \frac{\eta}{\gamma}\right)e^{2\gamma t} - \frac{\eta}{\gamma}.
$$

Thus, (2.2) holds for $p = 2$. Thanks to Hölder's inequality, (2.2) is also valid for $p \in (0;2]$.

Now, we suppose that (2.2) holds for all even integer $q \in [0, p-2]$ with even integer p, i.e.,

$$
\mathbb{E}\left[X_t^q\right] \le \begin{cases} C_q(1+e^{\gamma q t}) & \text{if } \gamma \neq 0, \\ C_q(1+t)^{q/2} & \text{if } \gamma = 0. \end{cases}
$$

We shall show that (2.2) holds for p. For this, using the binomial theorem, we have

$$\begin{aligned}
&(X_s + c(X_s)z)^p - X_s^p - pX_s^{p-1}c(X_s)z \\
&= \binom{p}{2} X_s^{p-2} c^2(X_s) z^2 + \sum_{i=3}^{p} \binom{p}{i} X_s^{p-i} c^i(X_s) z^i. \qquad (2.6)
\end{aligned}$$

For all $3 \le i \le p$, using Condition **B1**, the binomial theorem and the fact that $|x|^{p-3} \le \frac{1}{2}(|x|^{p-2} + |x|^{p-4})$ valid for any $x \in \mathbb{R}$, we get that

$$\begin{aligned}
& X_s^{p-i} c^i(X_s) \\
&= X_s^{p-i} c^2(X_s) c^{i-2}(X_s) \\
&\le |X_s|^{p-i} c^2(X_s) L_0^{i-2} (1 + |X_s|)^{i-2} \\
&= |X_s|^{p-i} c^2(X_s) L_0^{i-2} \left(|X_s|^{i-2} + (i-2)|X_s|^{i-3} + \sum_{j=2}^{i-2} \binom{i-2}{j} |X_s|^{i-2-j} \right) \\
&= c^2(X_s) L_0^{i-2} \left(|X_s|^{p-2} + (i-2)|X_s|^{p-3} + \sum_{j=2}^{i-2} \binom{i-2}{j} |X_s|^{p-2-j} \right) \\
&\le c^2(X_s) L_0^{i-2} \left(|X_s|^{p-2} + (i-2)\frac{1}{2} \left(|X_s|^{p-2} + |X_s|^{p-4}\right) + \sum_{j=2}^{i-2} \binom{i-2}{j} |X_s|^{p-2-j} \right) \\
&= c^2(X_s) L_0^{i-2} \left(\frac{i}{2}|X_s|^{p-2} + \frac{i-2}{2}|X_s|^{p-4} + \sum_{j=2}^{i-2} \binom{i-2}{j} |X_s|^{p-2-j} \right).
\end{aligned}$$

This, combined with the fact that $\sum_{i=2}^{p} \binom{p}{i} i a^i = pa\left((1+a)^{p-1} - 1\right)$ valid for any $a \in \mathbb{R}$, we obtain that

$$\begin{aligned}
& (X_s + c(X_s)z)^p - X_s^p - pX_s^{p-1}c(X_s)z \\
&\le \binom{p}{2} |X_s|^{p-2} c^2(X_s) z^2 + c^2(X_s)|X_s|^{p-2} \sum_{i=3}^{p} \binom{p}{i} L_0^{i-2} \frac{i}{2} |z|^i \\
&\quad + c^2(X_s) \sum_{i=3}^{p} \binom{p}{i} L_0^{i-2} \left(\frac{i-2}{2}|X_s|^{p-4} + \sum_{j=2}^{i-2} \binom{i-2}{j} |X_s|^{p-2-j} \right) |z|^i \\
&= c^2(X_s)|X_s|^{p-2} \frac{1}{2L_0^2} \sum_{i=2}^{p} \binom{p}{i} i (L_0|z|)^i \\
&\quad + c^2(X_s) \sum_{i=3}^{p} \binom{p}{i} L_0^{i-2} \left(\frac{i-2}{2}|X_s|^{p-4} + \sum_{j=2}^{i-2} \binom{i-2}{j} |X_s|^{p-2-j} \right) |z|^i \\
&= c^2(X_s)|X_s|^{p-2} \frac{p}{2L_0} |z| \left((1 + L_0|z|)^{p-1} - 1 \right)
\end{aligned}$$

$$+ c^2(X_s) \sum_{i=3}^{p} \binom{p}{i} L_0^{i-2} \left(\frac{i-2}{2} |X_s|^{p-4} + \sum_{j=2}^{i-2} \binom{i-2}{j} |X_s|^{p-2-j} \right) |z|^i. \tag{2.7}$$

Therefore, inserting (2.7) into (2.3), using Condition **B2**, Remark 2.3.2 and $c^2(x) \leq 2L_0^2(1+x^2)$ for any $x \in \mathbb{R}$, we get

$$\begin{aligned}
e^{-p\gamma t} X_t^p &\leq x_0^p + p\int_0^t e^{-p\gamma s} X_s^{p-2} \Big[-\gamma X_s^2 + X_s b(X_s) + \frac{p-1}{2}\sigma^2(X_s) \\
&\qquad + \frac{c^2(X_s)}{2L_0} \int_{\mathbb{R}_0} |z| \left((1+L_0|z|)^{p-1} - 1 \right) \nu(dz) \Big] ds \\
&\quad + \int_0^t e^{-p\gamma s} c^2(X_s) \sum_{i=3}^{p} \binom{p}{i} L_0^{i-2} \left[\frac{i-2}{2} |X_s|^{p-4} + \sum_{j=2}^{i-2} \binom{i-2}{j} |X_s|^{p-2-j} \right] \\
&\quad \times \int_{\mathbb{R}_0} |z|^i \nu(dz) ds + p\int_0^t e^{-p\gamma s} X_s^{p-1} \sigma(X_s) dW_s \\
&\quad + \int_0^t \int_{\mathbb{R}_0} e^{-p\gamma s} \left[(X_{s-} + c(X_{s-})z)^p - X_{s-}^p \right] \widetilde{N}(ds, dz) \\
&\leq x_0^p + p\eta \int_0^t e^{-p\gamma s} X_s^{p-2} ds \\
&\quad + 2\int_0^t e^{-p\gamma s} \sum_{i=3}^{p} \binom{p}{i} L_0^i \Big[\frac{i-2}{2} \left(|X_s|^{p-4} + |X_s|^{p-2} \right) \\
&\quad + \sum_{j=2}^{i-2} \binom{i-2}{j} \left(|X_s|^{p-2-j} + |X_s|^{p-j} \right) \Big] \int_{\mathbb{R}_0} |z|^i \nu(dz) ds \\
&\quad + p\int_0^t e^{-p\gamma s} X_s^{p-1} \sigma(X_s) dW_s \\
&\quad + \int_0^t \int_{\mathbb{R}_0} e^{-p\gamma s} \left[(X_{s-} + c(X_{s-})z)^p - X_{s-}^p \right] \widetilde{N}(ds, dz).
\end{aligned} \tag{2.8}$$

Now, it suffices to use the same argument as in the case $p = 2$ by replacing t by $t \wedge \tau_N$ in (2.8) and taking expectations on both sides to get that

$$\begin{aligned}
&\mathbb{E}\left[e^{-p\gamma(t\wedge\tau_N)} X_{t\wedge\tau_N}^p \right] \\
&\leq x_0^p + p\eta \int_0^t e^{-p\gamma s} \mathbb{E}\left[X_s^{p-2} \right] ds \\
&\quad + 2\int_0^t e^{-p\gamma s} \sum_{i=3}^{p} \binom{p}{i} L_0^i \Big[\frac{i-2}{2} \left(\mathbb{E}\left[|X_s|^{p-4} \right] + \mathbb{E}\left[|X_s|^{p-2} \right] \right) \\
&\quad + \sum_{j=2}^{i-2} \binom{i-2}{j} \left(\mathbb{E}\left[|X_s|^{p-2-j} \right] + \mathbb{E}\left[|X_s|^{p-j} \right] \right) \Big] \int_{\mathbb{R}_0} |z|^i \nu(dz) ds.
\end{aligned} \tag{2.9}$$

Then, using the fact that $\tau_N \uparrow \infty$ a.s. as $N \uparrow \infty$, letting $N \uparrow \infty$ on the left hand side of (2.9) and using Fatou's lemma and **B6**, we get

$$\mathbb{E}\left[e^{-p\gamma t} X_t^p\right] \leq x_0^p + p\eta \int_0^t e^{-p\gamma s} \mathbb{E}\left[X_s^{p-2}\right] ds + C \int_0^t e^{-p\gamma s} \sum_{i=2}^p \mathbb{E}\left[|X_s|^{p-i}\right] ds.$$

Now, it suffices to use the inductive assumption.

<u>If $\gamma = 0$:</u>

$$\begin{aligned}\mathbb{E}\left[X_t^p\right] &\leq x_0^p + C_p \int_0^t (1+s)^{p/2-1} ds + C_p \sum_{i=2}^p \int_0^t (1+s)^{(p-i)/2} ds \\ &\leq C_p (1+t)^{p/2}.\end{aligned}$$

<u>If $\gamma \neq 0$:</u>

$$\begin{aligned}\mathbb{E}\left[e^{-p\gamma t} X_t^p\right] &\leq x_0^p + C_p \int_0^t e^{-p\gamma s} \left(1 + e^{\gamma(p-2)s}\right) ds + C_p \sum_{i=2}^p \int_0^t e^{-p\gamma s} \left(1 + e^{\gamma(p-i)s}\right) ds \\ &\leq C_p + C_p e^{-p\gamma t}.\end{aligned}$$

This implies that

$$\mathbb{E}\left[X_t^p\right] \leq C_p + C_p e^{\gamma p t}.$$

Therefore, (2.2) is valid for p. By the induction principle, (2.2) holds for any even natural number $p \in [2, p_0]$. Finally, using Hölder's inequality, we finish the proof for any $p \in (0, p_0]$. □

Subsequently, we establish the existence and uniqueness of the solution for the equation (2.1).

Theorem 2.3.3. *Assume that the coefficients b, c and σ satisfy the conditions **B1**–**B5**. Assume further that the Lévy measure satisfies $\int_{\mathbb{R}_0} |z|\nu(dz) < \infty$ and $\int_{\mathbb{R}_0} z^2\nu(dz) < \infty$. Then the path-wise uniqueness holds for equation (2.1).*

Moreover, suppose that there exist positive constants C and $\ell \in (0; \frac{p_0}{4}]$ such that

$$|b(x)| \vee |\sigma(x)| \vee |c(x)| \leq C\left(1 + |x|^\ell\right),$$

*for all $x \in \mathbb{R}$, where p_0 is defined in Condition **B2**. Then the equation (2.1) has a strong solution.*

Proof. Without loss of generality, we assume that γ is a positive constant.

Existence of solution:

For each $N > 0$, set

$$b_N(x) = \begin{cases} b(x), & \text{if } |x| \leq N, \\ b\left(\frac{Nx}{|x|}\right)(N+1-|x|), & \text{if } N < |x| < N+1, \\ 0, & \text{if } |x| \geq N+1, \end{cases}$$

and

$$\sigma_N(x) = \begin{cases} \sigma(x), & \text{if } |x| \leq N, \\ \sigma\left(\frac{Nx}{|x|}\right)(N+1-|x|), & \text{if } N < |x| < N+1, \\ 0, & \text{if } |x| \geq N+1, \end{cases}$$

and

$$c_N(x) = \begin{cases} c(x), & \text{if } |x| \leq N, \\ c\left(\frac{Nx}{|x|}\right)(N+1-|x|), & \text{if } N < |x| < N+1, \\ 0, & \text{if } |x| \geq N+1. \end{cases}$$

It is clear that b_N, c_N and σ_N satisfy Assumptions of Theorem 2.2 in [58]. Thus, the equation

$$X_t^N = x_0 + \int_0^t b_N\left(X_s^N\right) ds + \int_0^t \sigma_N\left(X_s^N\right) dW_s + \int_0^t \int_{\mathbb{R}_0} c_N\left(X_{s-}^N\right) z\widetilde{N}(ds, dz) \tag{2.10}$$

has a unique strong solution X_t^N. We will show that when $N \to \infty$, X_t^N converges in probability to a process X_t which satisfies equation (2.1).

For each $N > 0$, set

$$\tau_N = T \wedge \inf\left\{t \in [0;T] : \left|X_t^N\right| \geq N\right\}.$$

Due to the pathwise uniqueness of solution to equation (2.10), $X_t^N = X_t^M$ almost surely for any $t < \tau_N$ and $N < M$. Then, we will show that $\tau_N = T$ almost surely for all N large enough.

In addition, it is straightforward to show that the coefficients $b_N(x)$, $c_N(x)$ and $\sigma_N(x)$ satisfy Condition **B2'**:

$$p_0 x b_N(x) + \frac{p_0(p_0-1)}{2}\sigma_N^2(x) + \frac{c_N^2(x)}{4L_0^2}\int_{\mathbb{R}_0}\left((1+2L_0|z|)^{p_0} - 1 - 2p_0L_0|z|\right)\nu(dz)$$

$$\leq 2\gamma|x|^2 + 2\eta,$$

for any $x \in \mathbb{R}$.

Firstly, by applying Itô's formula for $|X_t^N|^2$ and Condition **B2'** with $p_0 = 2$, we have

$$\begin{aligned}
|X_t^N|^2 &= x_0^2 + \int_0^t \left(2X_s^N b_N\left(X_s^N\right) + \sigma_N^2\left(X_s^N\right)\right) ds + \int_0^t 2X_s^N \sigma_N\left(X_s^N\right) dW_s \\
&\quad + \int_0^t \int_{\mathbb{R}_0} \left(\left(X_s^N + c_N\left(X_s^N\right) z\right)^2 - |X_s^N|^2 - 2X_s^N c_N\left(X_s^N\right) z\right) \nu(dz) ds \\
&\quad + \int_0^t \int_{\mathbb{R}_0} \left(\left(X_{s-}^N + c_N\left(X_{s-}^N\right) z\right)^2 - |X_{s-}^N|^2\right) \widetilde{N}(ds, dz) \\
&= x_0^2 + \int_0^t \left(2X_s^N b_N\left(X_s^N\right) + \sigma_N^2\left(X_s^N\right)\right) ds + \int_0^t 2X_s^N \sigma_N\left(X_s^N\right) dW_s \\
&\quad + \int_0^t \int_{\mathbb{R}_0} c_N^2\left(X_s^N\right) z^2 \nu(dz)\, ds + \int_0^t \int_{\mathbb{R}_0} \left(2X_{s-}^N c_N\left(X_{s-}^N\right) z + c_N^2\left(X_{s-}^N\right) z^2\right) \widetilde{N}(ds, dz) \\
&= x_0^2 + \int_0^t \left(2X_s^N b_N\left(X_s^N\right) + \sigma_N^2\left(X_s^N\right) + c_N^2\left(X_s^N\right) \int_{\mathbb{R}_0} z^2 \nu(dz)\right) ds \\
&\quad + \int_0^t 2X_s^N \sigma_N\left(X_s^N\right) dW_s + \int_0^t \int_{\mathbb{R}_0} \left(2X_{s-}^N c_N\left(X_{s-}^N\right) z + c_N^2\left(X_{s-}^N\right) z^2\right) \widetilde{N}(ds, dz) \\
&\leq x_0^2 + 2\int_0^t \left(\gamma\left|X_s^N\right|^2 + \eta\right) ds + \int_0^t 2X_s^N \sigma_N\left(X_s^N\right) dW_s \\
&\quad + \int_0^t \int_{\mathbb{R}_0} \left(2X_{s-}^N c_N\left(X_{s-}^N\right) z + c_N^2\left(X_{s-}^N\right) z^2\right) \widetilde{N}(ds, dz), \qquad (2.11)
\end{aligned}$$

Taking expectation on both sides (2.11) to get

$$\begin{aligned}
\mathbb{E}\left[|X_t^N|^2\right] &\leq x_0^2 + 2\mathbb{E}\left[\int_0^t \left(\gamma\left|X_s^N\right|^2 + \eta\right) ds\right] \\
&\leq x_0^2 + 2\int_0^T \left(\gamma\mathbb{E}\left[\left|X_s^N\right|^2\right] + \eta\right) ds \\
&\leq (x_0^2 + 2\eta T) + 2|\gamma| \int_0^T \mathbb{E}\left[\left|X_s^N\right|^2\right] ds.
\end{aligned}$$

By using the Gronwall's inequality, we have

$$\mathbb{E}\left[|X_t^N|^2\right] \leq (x_0^2 + 2\eta T)e^{2|\gamma|T} = C(x_0, \gamma, \eta, T). \qquad (2.12)$$

On the other hand, the estimate (2.11) implies that

$$|X_{t\wedge\tau}^N|^2 \leq x_0^2 + 2\int_0^{t\wedge\tau} \left(\gamma\left|X_s^N\right|^2 + \eta\right) ds + \int_0^{t\wedge\tau} 2X_s^N \sigma_N\left(X_s^N\right) dW_s$$

$$+\int_0^{t\wedge\tau}\int_{\mathbb{R}_0}\left(2X_{s-}^N c_N\left(X_{s-}^N\right)z+c_N^2\left(X_{s-}^N\right)z^2\right)\widetilde{N}(ds,dz), \tag{2.13}$$

for any stopping time $\tau \leq T$.

Taking expectation on both sides (2.13) and using (2.12) to obtain

$$\begin{aligned}\mathbb{E}\left[|X_{t\wedge\tau}^N|^2\right] &\leq x_0^2+2\mathbb{E}\left[\int_0^{t\wedge\tau}\left(\gamma\left|X_s^N\right|^2+\eta\right)ds\right]\\ &\leq x_0^2+2\int_0^T\left(|\gamma|\mathbb{E}\left[\left|X_s^N\right|^2\right]+\eta\right)ds\\ &\leq C(x_0,\gamma,\eta,T).\end{aligned}$$

For any $p\in(0;2)$, thanks to Proposition IV.4.7 in [78], we have

$$\mathbb{E}\left[\sup_{0\leq t\leq T}\left|X_t^N\right|^p\right]\leq C(x_0,\gamma,\eta,p,T)\quad\text{for any } N>0. \tag{2.14}$$

Thus,

$$C\geq\mathbb{E}\left[\sup_{0\leq t\leq T}\left|X_t^N\right|^p\right]\geq\mathbb{E}\left[\sup_{0\leq t\leq T}\left|X_t^N\right|^p\mathbb{I}_{[\tau_N<T]}\right]\geq N^p\mathbb{P}\left[\tau_N<T\right].$$

It leads to $\sum_{N=1}^{\infty}\mathbb{P}\left(\tau_N<T\right)<\infty$. Thanks to Borel-Cantelli's lemma, we obtain

$$\mathbb{P}\left[\limsup_N\left\{\tau_N<T\right\}\right]=0.$$

Since $(\tau_N)_N$ is increasing, $\tau_N=T$ for all N large enough. It means $\lim_{N\to\infty}X_t^N=X_t$ exists almost surely and $X_t=X_t^M$ almost surely for any $t<\tau_N$ and $M\geq N$.

Secondly, for any $\kappa>0$, $k>1$, $2<q\leq p_0$,

$$\begin{aligned}\mathbb{E}\left[\left|X_{t\wedge\tau_{N+k}}^{N+k}-X_{t\wedge\tau_N}^N\right|^2\right]&\leq 2\mathbb{E}\left[\left(\left|X_t^{N+k}\right|^2+\left|X_t^N\right|^2\right)\mathbb{I}_{\{\tau_N<T\}}\right]\\ &\leq C_q\left(\kappa\mathbb{E}\left[\left|X_t^{N+k}\right|^q\right]+\kappa\mathbb{E}\left[\left|X_t^N\right|^q\right]+\frac{\mathbb{P}\left[\tau_N<T\right]}{\kappa^{2/(q-2)}}\right).\end{aligned}$$

Let $N\to\infty$ and then let $\kappa\to 0$, we get

$$\mathbb{E}\left[\left|X_{t\wedge\tau_{N+M}}^{N+M}-X_{t\wedge\tau_N}^N\right|^2\right]\to 0\quad\text{as } N\to\infty. \tag{2.15}$$

It means that $(X_{t\wedge\tau_N}^N)_{N\geq 1}$ is a Cauchy sequence in L^2 space. This implies

$$X_{t\wedge\tau_N}^N\xrightarrow{L^2}X_t\quad\text{as } N\to\infty.$$

Since coefficients b_N, c_N and σ_N satisfy Condition **B2'**, it follows that they also satisfy Condition **B2**. Thus, from Proposition 2.3.1, we have that for any $p\in(0,p_0]$, there exists a constant $C_p > 0$ such that

$$\sup_{0\le t\le T} \mathbb{E}\left[|X_t^N|^p\right] \le C_p.$$

Thanks to Fatou's lemma, there exists a constant $C_p > 0$ such that

$$\sup_{0\le t\le T} \mathbb{E}\left[|X_t|^p\right] \le C_p.$$

From the definition of $b_N(x)$, we have

$$\mathbb{E}\left[\left|\int_0^{t\wedge\tau_N} \left[b_N\left(X_s^N\right) - b(X_s)\right] ds\right|^2\right] = 0.$$

Moreover, using the fact that $|b(x)| \le C\left(1+|x|^\ell\right)$ for all $x \in \mathbb{R}$ and $p_0 \ge 4l$,

$$\begin{aligned}
\mathbb{E}\left[\left|\int_{t\wedge\tau_N}^t b(X_s)ds\right|^2\right] &\le C\int_0^t \mathbb{E}\left[\left(1+|X_s|^{2\ell}\right)\mathbb{I}_{\{\tau_N\le s\}}\right] ds \\
&\le \kappa C\int_0^T \mathbb{E}\left[1+|X_s|^{2\ell}\right]^2 ds + \frac{C\mathbb{P}\left[\tau_N < T\right]}{\kappa} \\
&\le \kappa C + \frac{C\mathbb{P}\left[\tau_N < T\right]}{\kappa}.
\end{aligned}$$

Let $N \to \infty$ and $\kappa \to 0$, we have

$$\int_0^{t\wedge\tau_N} b_N\left(X_s^N\right) ds \xrightarrow{L^2} \int_0^t b(X_s)ds \quad \text{as } N\to\infty. \tag{2.16}$$

In the same manner, we can see that

$$\int_0^{t\wedge\tau_N} \sigma_N\left(X_s^N\right) dW_s \xrightarrow{L^2} \int_0^t \sigma(X_s)dW_s \quad \text{as } N\to\infty, \tag{2.17}$$

and, by $\int_{\mathbb{R}_0} z^2\nu(dz) < \infty$,

$$\int_0^{t\wedge\tau_N}\int_{\mathbb{R}_0} c_N\left(X_{s-}^N\right) z\widetilde{N}(ds,dz) \xrightarrow{L^2} \int_0^t\int_{\mathbb{R}_0} c\left(X_{s-}\right) z\widetilde{N}(ds,dz) \quad \text{as } N\to\infty. \tag{2.18}$$

Finally, by combining (2.10), (2.15), (2.16), (2.17) and (2.18), we get

$$X_t = x_0 + \int_0^t b(X_s)ds + \int_0^t \sigma(X_s)dW_s + \int_0^t\int_{\mathbb{R}_0} c\left(X_{s-}\right) z\widetilde{N}(ds,dz)$$

almost surely for all $t \in [0;T]$. This shows that $(X_t)_{t\in[0;T]}$ is a solution of equation (2.1).

Pathwise uniqueness:

Suppose that equation (2.1) has a solution $(X_t)_{0\le t\le T}$, and $(X'_t)_{0\le t\le T}$ is another solution of equation (2.1). It follows from the proof of Proposition 2.3.1 that the sample paths of $(X_t)_{0\le t\le T}$ and $(X'_t)_{0\le t\le T}$ do not explode. We are going to show that $\mathbb{E}\left[|X_t - X'_t|\right] = 0$ for all $t \in [0;T]$, which implies the uniqueness of the solution. For each $N > 0$, let $\tau_N = T \wedge \inf\{t \ge 0 : |X_t| \vee |X'_t| \ge N\}$.

Firstly, applying Itô's formula for X_t^2 and Condition **B2** with $p_0 = 2$, we have

$$\begin{aligned}
X_t^2 &= x_0^2 + \int_0^t \left(2X_s b(X_s) + \sigma^2(X_s)\right) ds + \int_0^t 2X_s\sigma(X_s)\, dW_s \\
&\quad + \int_0^t \int_{\mathbb{R}_0} \left((X_s + c(X_s) z)^2 - X_s^2 - 2X_s c(X_s) z\right) \nu(dz)ds \\
&\quad + \int_0^t \int_{\mathbb{R}_0} \left((X_{s-} + c(X_{s-}) z)^2 - X_{s-}^2\right) \widetilde{N}(ds, dz) \\
&= x_0^2 + \int_0^t \left(2X_s b(X_s) + \sigma^2(X_s)\right) ds + \int_0^t 2X_s\sigma(X_s)\, dW_s \\
&\quad + \int_0^t \int_{\mathbb{R}_0} c^2(X_s) z^2 \nu(dz)\, ds + \int_0^t \int_{\mathbb{R}_0} \left(2X_{s-}c(X_{s-}) z + c^2(X_{s-}) z^2\right) \widetilde{N}(ds, dz) \\
&= x_0^2 + \int_0^t \left(2X_s b(X_s) + \sigma^2(X_s) + c^2(X_s) \int_{\mathbb{R}_0} z^2 \nu(dz)\right) ds \\
&\quad + \int_0^t 2X_s\sigma(X_s)\, dW_s + \int_0^t \int_{\mathbb{R}_0} \left(2X_{s-}c(X_{s-}) z + c^2(X_{s-}) z^2\right) \widetilde{N}(ds, dz) \\
&\le x_0^2 + 2\int_0^t \left(\gamma |X_s|^2 + \eta\right) ds + \int_0^t 2X_s\sigma(X_s)\, dW_s \\
&\quad + \int_0^t \int_{\mathbb{R}_0} \left(2X_{s-}c(X_{s-}) z + c^2(X_{s-}) z^2\right) \widetilde{N}(ds, dz),
\end{aligned}$$

which implies

$$\begin{aligned}
X_{t\wedge\tau_N}^2 &\le x_0^2 + 2\int_0^{t\wedge\tau_N} \left(\gamma |X_s|^2 + \eta\right) ds + \int_0^{t\wedge\tau_N} 2X_s\sigma(X_s)\, dW_s \\
&\quad + \int_0^{t\wedge\tau_N} \int_{\mathbb{R}_0} \left(2X_{s-}c(X_{s-}) z + c^2(X_{s-}) z^2\right) \widetilde{N}(ds, dz).
\end{aligned}$$

For any stopping time $\tau \le T$, we get

$$\begin{aligned}
X_{\tau\wedge\tau_N}^2 &\le x_0^2 + 2\int_0^{\tau\wedge\tau_N} \left(\gamma |X_s|^2 + \eta\right) ds + \int_0^{\tau\wedge\tau_N} 2X_s\sigma(X_s)\, dW_s \\
&\quad + \int_0^{\tau\wedge\tau_N} \int_{\mathbb{R}_0} \left(2X_{s-}c(X_{s-}) z + c^2(X_{s-}) z^2\right) \widetilde{N}(ds, dz). \qquad (2.19)
\end{aligned}$$

Taking expectation on both sides (2.19) and using Proposition 2.3.1 to obtain

$$\begin{aligned}\mathbb{E}\left[X^2_{\tau\wedge\tau_N}\right] &\leq x_0^2 + 2\mathbb{E}\left[\int_0^{\tau\wedge\tau_N}\left(\gamma\left|X_s\right|^2+\eta\right)ds\right]\\ &\leq x_0^2 + 2\int_0^T\left(\gamma\mathbb{E}\left[\left|X_s\right|^2\right]+\eta\right)ds\\ &\leq C(x_0,\gamma,\eta,T).\end{aligned}$$

For any $p\in(0;2)$, thanks to Proposition IV.4.7 in [78], we have

$$\mathbb{E}\left[\sup_{0\leq t\leq T}\left|X_{t\wedge\tau_N}\right|^p\right]\leq C(x_0,\gamma,\eta,p,T).$$

Let $N\to\infty$ and apply Fatou's lemma, we get

$$\mathbb{E}\left[\sup_{0\leq t\leq T}\left|X_t\right|^p\right]\leq C(x_0,\gamma,\eta,p,T). \tag{2.20}$$

Similarly, since $(X'_t)_{0\leq t\leq T}$ is another solution of equation (2.1), as in (2.20), for any $p\in(0;2)$, we have

$$\mathbb{E}\left[\sup_{0\leq t\leq T}\left|X'_t\right|^p\right]\leq C\left(x_0,\gamma,\eta,p,T\right). \tag{2.21}$$

Since X_t and X'_t are solutions of (2.1), we write

$$\begin{aligned}X_t - X'_t = &\int_0^t\left[b\left(X_s\right)-b\left(X'_s\right)\right]ds + \int_0^t\left[\sigma\left(X_s\right)-\sigma\left(X'_s\right)\right]dW_s\\ &+\int_0^t\int_{\mathbb{R}_0}\left(c\left(X_{s-}\right)-c\left(X'_{s-}\right)\right)z\widetilde{N}\left(ds,dz\right).\end{aligned}$$

Applying Itô's formula for $\phi_{\delta\varepsilon}\left(X_t-X'_t\right)$ and using the mean value theorem, **YW2** and **YW5**, we get

$$\begin{aligned}\left|X_{t\wedge\tau_N}-X'_{t\wedge\tau_N}\right| \leq\ & \varepsilon+\phi_{\delta\varepsilon}\left(X_{t\wedge\tau_N}-X'_{t\wedge\tau_N}\right)\\ =\ & \varepsilon+\int_0^{t\wedge\tau_N}\phi'_{\delta\varepsilon}\left(X_s-X'_s\right)\left(b\left(X_s\right)-b\left(X'_s\right)\right)ds\\ &+\int_0^{t\wedge\tau_N}\frac{1}{2}\phi''_{\delta\varepsilon}\left(X_s-X'_s\right)\left(\sigma\left(X_s\right)-\sigma\left(X'_s\right)\right)^2ds\\ &+\int_0^{t\wedge\tau_N}\phi'_{\delta\varepsilon}\left(X_s-X'_s\right)\left(\sigma\left(X_s\right)-\sigma\left(X'_s\right)\right)dW_s\\ &+\int_0^{t\wedge\tau_N}\int_{\mathbb{R}_0}\left[\phi_{\delta\varepsilon}\left(X_s-X'_s+\left(c\left(X_s\right)-c\left(X'_s\right)\right)z\right)\right.\end{aligned}$$

$$
\begin{aligned}
&- \phi_{\delta\varepsilon}\left(X_s - X_s'\right) - \phi_{\delta\varepsilon}'\left(X_s - X_s'\right)\left(c\left(X_s\right) - c\left(X_s'\right)z\right)\Big]\nu(dz)ds \\
&+ \int_0^{t\wedge\tau_N}\int_{\mathbb{R}_0}\Big[\phi_{\delta\varepsilon}\left(X_{s-} - X_{s-}' + \left(c\left(X_{s-}\right) - c\left(X_{s-}'\right)\right)z\right) \\
&- \phi_{\delta\varepsilon}\left(X_{s-} - X_{s-}'\right)\Big]\widetilde{N}\left(ds, dz\right) \\
\leq{}& \varepsilon + \int_0^{t\wedge\tau_N}\left|b\left(X_s\right) - b\left(X_s'\right)\right| ds \\
&+ \int_0^{t\wedge\tau_N}\frac{1}{\left|X_s - X_s'\right|\log\delta}\mathbb{I}_{\left[\frac{\varepsilon}{\delta},\varepsilon\right]}\left(\left|X_s - X_s'\right|\right)\left(\sigma\left(X_s\right) - \sigma\left(X_s'\right)\right)^2 ds \\
&+ \int_0^{t\wedge\tau_N}\phi_{\delta\varepsilon}'\left(X_s - X_s'\right)\left(\sigma\left(X_s\right) - \sigma\left(X_s'\right)\right)dW_s \\
&+ \int_0^{t\wedge\tau_N}\int_{\mathbb{R}_0} 2\left|c\left(X_s\right) - c\left(X_s'\right)\right| |z|\nu\left(dz\right)ds \\
&+ \int_0^{t\wedge\tau_N}\int_{\mathbb{R}_0}\Big[\phi_{\delta\varepsilon}\left(X_{s-} - X_{s-}' + \left(c\left(X_{s-}\right) - c\left(X_{s-}'\right)\right)z\right) \\
&- \phi_{\delta\varepsilon}\left(X_{s-} - X_{s-}'\right)\Big]\widetilde{N}\left(ds, dz\right).
\end{aligned}
$$

Then, using Conditions **B3, B4, B5**, we get

$$
\begin{aligned}
\left|X_{t\wedge\tau_N} - X_{t\wedge\tau_N}'\right| \leq{}& \varepsilon + \int_0^{t\wedge\tau_N} L_N\left|X_s - X_s'\right| ds + \int_0^{t\wedge\tau_N}\frac{\varepsilon^{2\alpha}L_N^2}{\log\delta}ds \\
&+ \int_0^{t\wedge\tau_N}\phi_{\delta\varepsilon}'\left(X_s - X_s'\right)\left(\sigma\left(X_s\right) - \sigma\left(X_s'\right)\right)dW_s \\
&+ \int_0^{t\wedge\tau_N}\int_{\mathbb{R}_0} 2L_N\left|X_s - X_s'\right| |z|\nu(dz)ds \\
&+ \int_0^{t\wedge\tau_N}\int_{\mathbb{R}_0}\Big[\phi_{\delta\varepsilon}\left(X_{s-} - X_{s-}' + \left(c\left(X_{s-}\right) - c\left(X_{s-}'\right)\right)z\right) \\
&- \phi_{\delta\varepsilon}\left(X_{s-} - X_{s-}'\right)\Big]\widetilde{N}\left(ds, dz\right) \\
\leq{}& \varepsilon + \int_0^{t\wedge\tau_N} L_N\left(1 + 2\mu\right)\left|X_s - X_s'\right| ds + \int_0^{t\wedge\tau_N}\frac{\varepsilon^{2\alpha}L_N^2}{\log\delta}ds \\
&+ \int_0^{t\wedge\tau_N}\phi_{\delta\varepsilon}'\left(X_s - X_s'\right)\left(\sigma\left(X_s\right) - \sigma\left(X_s'\right)\right)dW_s \\
&+ \int_0^{t\wedge\tau_N}\int_{\mathbb{R}_0}\Big[\phi_{\delta\varepsilon}\left(X_{s-} - X_{s-}' + \left(c\left(X_{s-}\right) - c\left(X_{s-}'\right)\right)z\right) \\
&- \phi_{\delta\varepsilon}\left(X_{s-} - X_{s-}'\right)\Big]\widetilde{N}\left(ds, dz\right).
\end{aligned}
\tag{2.22}
$$

By taking expectation on both sides (2.22) and using Proposition 2.3.1, we obtain

$$\mathbb{E}\left[|X_{t\wedge\tau_N} - X'_{t\wedge\tau_N}\right] \leq \varepsilon + L_N\left(1+2\mu\right)\int_0^t \mathbb{E}\left[\left|X_{s\wedge\tau_N} - X'_{s\wedge\tau_N}\right|\right] ds + \frac{\varepsilon^{2\alpha}L_N^2 T}{\log\delta}.$$

By choosing $\delta = 2$ and letting $\varepsilon \to 0$, we get

$$\mathbb{E}\left[\left|X_{t\wedge\tau_N} - X'_{t\wedge\tau_N}\right|\right] \leq L_N(1+2\mu)\int_0^t \mathbb{E}\left[\left|X_{s\wedge\tau_N} - X'_{s\wedge\tau_N}\right|\right] ds.$$

Thanks to Gronwall's inequality, $\mathbb{E}\left[\left|X_{t\wedge\tau_N} - X'_{t\wedge\tau_N}\right|\right] = 0$. It means $X_{t\wedge\tau_N} = X'_{t\wedge\tau_N}$ almost surely. This leads to $\mathbb{E}\left[|X_t - X'_t|\right] = \mathbb{E}\left[|X_t - X'_t|\,\mathbb{I}_{[\tau_N \leq t]}\right]$. By applying Cauchy's inequality, (2.20) and (2.21) for any $p \in (1;2)$, we obtain

$$\begin{aligned}
\mathbb{E}\left[\left|X_t - X'_t\right|\right] &\leq \frac{1}{2N}\mathbb{E}\left[\left|X_t - X'_t\right|^2\right] + \frac{N}{2}\mathbb{P}\left[\tau_N \leq T\right] \\
&\leq \frac{1}{2N}\mathbb{E}\left[\left|X_t - X'_t\right|^2\right] + \frac{1}{2N^{p-1}}\left(\mathbb{E}\left[\sup_{0\leq t\leq T}|X_t|^p\right] + \mathbb{E}\left[\sup_{0\leq t\leq T}\left|X'_t\right|^p\right]\right) \\
&\leq \frac{1}{2N}\mathbb{E}\left[\left|X_t - X'_t\right|^2\right] + \frac{C}{2N^{p-1}}.
\end{aligned}$$

Let $N \to \infty$ we obtain $\mathbb{E}\left[|X_t - X'_t|\right] = 0$. It means $X_t = X'_t$ almost surely for any $t \in [0;T]$. Since X and X' are càdlàg, they are indistinguishable on $[0,T]$. The proof is complete. □

2.4 Tamed-adaptive Euler-Maruyama scheme

In this section, we propose a new Euler-Maruyama approximation scheme for equation (2.1). This new scheme is constructed based on a combination of the tamed Euler-Maruyama and the adaptive Euler-Maruyama schemes, and we refer to it as the tamed-adaptive Euler-Maruyama scheme.

For each $\Delta \in (0,1)$, the tamed-adaptive Euler-Maruyama discretisation of equation (2.1) is defined as follows

$$\begin{cases} t_0 = 0, \quad \widehat{X}_0 = x_0, \quad t_{k+1} = t_k + h(\widehat{X}_{t_k})\Delta, \\ \widehat{X}_{t_{k+1}} = \widehat{X}_{t_k} + b(\widehat{X}_{t_k})h(\widehat{X}_{t_k})\Delta + \sigma_\Delta(\widehat{X}_{t_k})(W_{t_{k+1}} - W_{t_k}) + c_\Delta\left(\widehat{X}_{t_k}\right)\left(Z_{t_{k+1}} - Z_{t_k}\right), \end{cases} \tag{2.23}$$

where

$$h(x) = \frac{1}{(1+|b(x)|+|\sigma(x)|+|x|^l)^2 + |c(x)|^{p_0}}, \tag{2.24}$$

Here, $l \geq 1$ and $p_0 \geq 2$ are positive constants. The functions c_Δ and σ_Δ are approximations of c and σ, respectively, whose precise forms will be specified later.

As discussed in Chapter 1, the core idea of an adaptive scheme is to let the step size depend on the current state. Our scheme combines this with a taming approach: the step size $h_k := h(\widehat{X}_{t_k})\Delta$ is both adaptive (via $h(x)$) and scaled (via Δ), and we also employ tamed coefficients σ_Δ and c_Δ.

Proposition 2.4.1. *Suppose that there exist positive constants L and β such that the coefficients $b, c, \sigma, c_\Delta, \sigma_\Delta$ satisfy the following conditions*

T1. $|b(x)| \vee |\sigma(x)| \leq L\left(1+|x|^\beta\right)$*;*

T2. $x(b(x)-b(0)) \leq L|x|^2$*;*

T3. $|\sigma_\Delta(x)| \leq L|\sigma(x)|$ *and* $|c_\Delta(x)| \leq |c(x)|$*;*

T4. $|\sigma_\Delta(x)| \leq \dfrac{L}{\sqrt{\Delta}}$*;* $|c_\Delta(x)| \leq \dfrac{L}{\sqrt{\Delta}}$ *and* $|b(x)c_\Delta(x)| \leq \dfrac{L}{\sqrt{\Delta}}$*;*

for any $x \in \mathbb{R}$. Assume further that the Lévy measure satisfies $\int_{\mathbb{R}_0} z^2\nu(dz) < \infty$. Then

$$\lim_{k\to+\infty} t_k = +\infty \quad a.s.$$

Proof. Throughout the thesis, the following classical moment estimates for the increment of Brownian motion W and Lévy process Z will be useful

$$\mathbb{E}\left[\left|W_t - W_{\underline{t}^H}\right|^r \left|\mathcal{F}_{\underline{t}^H}\right.\right] \leq C_r\left(t-\underline{t}^H\right)^{r/2}; \quad \mathbb{E}\left[\left|Z_t - Z_{\underline{t}^H}\right|^r \left|\mathcal{F}_{\underline{t}^H}\right.\right] \leq C_r\left(t-\underline{t}^H\right), \tag{2.25}$$

for any $t > 0, r \geq 1$ and some positive constant C_r.

The proof strategy is based on the projection method outlined in [18]. This method involves introducing an auxiliary truncated (”projected”) process. By demonstrating that the moments of this auxiliary process are uniformly bounded (independently of the truncation level H), we can use Markov’s inequality to show that the probability of the original process coinciding with the truncated one tends to 1 as $H \to \infty$. Consequently, the two classes of approximate solutions coincide almost surely, and therefore share common properties. Moreover, to handle the irregular coefficients (specifically, the one-sided Lipschitz drift and Hölder continuous diffusion), we employ the Yamada-Watanabe function $\phi_{\delta\varepsilon}$ instead of the L^p-norm, which was used in [18].

For each $H > |x_0|$, a projected approximation scheme is defined as follows

$$\begin{cases} t_0^H = 0, \quad t_{k+1}^H = t_k^H + h(\widehat{X}_{t_k^H}^H)\Delta, \\ \widehat{X}_{t_{k+1}^H}^H = P_H\left(\widehat{X}_{t_k^H}^H + b\left(\widehat{X}_{t_k^H}^H\right)h\left(\widehat{X}_{t_k^H}^H\right)\Delta + \sigma_\Delta\left(\widehat{X}_{t_k^H}^H\right)(W_{t_{k+1}^H} - W_{t_k^H}) \right. \\ \qquad\qquad \left. + c_\Delta\left(\widehat{X}_{t_k^H}^H\right)(Z_{t_{k+1}^H} - Z_{t_k^H})\right), \end{cases}$$

where $P_H(Y) := \min\left(1, \frac{H}{|Y|}\right) Y$. Observe that $\left|\widehat{X}^H_{t^H_k}\right| \leq H$ for all k. Thus $h(\widehat{X}^H_{t^H_k})\Delta \geq C(H, L, l, m)\Delta$, which implies that $t^H_k \uparrow \infty$ as $k \to \infty$. We note that for each k, t^H_k is a stopping time and t^H_{k+1} is $\mathcal{F}_{t^H_k}$-measurable. Set $\underline{t}^H = \max\{t^H_k : t^H_k \leq t\}$. Then $\underline{t}^H$ is also a stopping time.

The continuous approximation is given by

$$\begin{aligned}\widehat{X}^H_t = P_H \Big(& \widehat{X}^H_{\underline{t}^H} + b\left(\widehat{X}^H_{\underline{t}^H}\right)(t - \underline{t}^H) + \sigma_\Delta\left(\widehat{X}^H_{\underline{t}^H}\right)\left(W_t - W_{\underline{t}^H}\right) \\ & + c_\Delta\left(\widehat{X}^H_{\underline{t}^H}\right)\left(Z_t - Z_{\underline{t}^H}\right)\Big).\end{aligned}$$

First, the fact that $P_H(Y) = \min(1, H/|Y|)Y$ implies $\phi_{\delta\varepsilon}(P_H(Y)) \leq \phi_{\delta\varepsilon}(Y)$. Therefore,

$$\begin{aligned}\phi_{\delta\varepsilon}\left(\widehat{X}^H_t\right) &= \phi_{\delta\varepsilon}\Big[P_H\Big(\widehat{X}^H_{\underline{t}^H} + b\left(\widehat{X}^H_{\underline{t}^H}\right)(t - \underline{t}^H) + \sigma_\Delta\left(\widehat{X}^H_{\underline{t}^H}\right)\left(W_t - W_{\underline{t}^H}\right) \\ &\qquad + c_\Delta\left(\widehat{X}^H_{\underline{t}^H}\right)\left(Z_t - Z_{\underline{t}^H}\right)\Big)\Big] \\ &\leq \phi_{\delta\varepsilon}\Big[\widehat{X}^H_{\underline{t}^H} + b\left(\widehat{X}^H_{\underline{t}^H}\right)(t - \underline{t}^H) + \sigma_\Delta\left(\widehat{X}^H_{\underline{t}^H}\right)\left(W_t - W_{\underline{t}^H}\right) \\ &\qquad + c_\Delta\left(\widehat{X}^H_{\underline{t}^H}\right)\left(Z_t - Z_{\underline{t}^H}\right)\Big].\end{aligned}$$

Next, using Taylor's expansion, there exists an $(\mathcal{F}_t)$-adapted process (ξ_t) such that

$$\begin{aligned}\phi_{\delta\varepsilon}\left(\widehat{X}^H_t\right) \leq \phi_{\delta\varepsilon}\left(\widehat{X}^H_{\underline{t}^H}\right) &+ \phi'_{\delta\varepsilon}\left(\widehat{X}^H_{\underline{t}^H}\right)\Big[b\left(\widehat{X}^H_{\underline{t}^H}\right)(t - \underline{t}^H) + \sigma_\Delta\left(\widehat{X}^H_{\underline{t}^H}\right)\left(W_t - W_{\underline{t}^H}\right) \\ &+ c_\Delta\left(\widehat{X}^H_{\underline{t}^H}\right)\left(Z_t - Z_{\underline{t}^H}\right)\Big] + \frac{1}{2}\phi''_{\delta\varepsilon}(\xi_t)\Big[b\left(\widehat{X}^H_{\underline{t}^H}\right)(t - \underline{t}^H) \\ &+ \sigma_\Delta\left(\widehat{X}^H_{\underline{t}^H}\right)\left(W_t - W_{\underline{t}^H}\right) + c_\Delta\left(\widehat{X}^H_{\underline{t}^H}\right)\left(Z_t - Z_{\underline{t}^H}\right)\Big]^2.\end{aligned}$$

Recall that $|\phi''_{\delta\varepsilon}(x)| \leq \frac{2\delta}{\varepsilon \log \delta}$. Thus,

$$\begin{aligned}\phi_{\delta\varepsilon}\left(\widehat{X}^H_t\right) \leq \phi_{\delta\varepsilon}\left(\widehat{X}^H_{\underline{t}^H}\right) &+ \phi'_{\delta\varepsilon}\left(\widehat{X}^H_{\underline{t}^H}\right)\Big[b\left(\widehat{X}^H_{\underline{t}^H}\right) - b(0)\Big](t - \underline{t}^H) \\ &+ \phi'_{\delta\varepsilon}\left(\widehat{X}^H_{\underline{t}^H}\right) b(0)(t - \underline{t}^H) + \phi'_{\delta\varepsilon}\left(\widehat{X}^H_{\underline{t}^H}\right)\sigma_\Delta\left(\widehat{X}^H_{\underline{t}^H}\right)\left(W_t - W_{\underline{t}^H}\right) \\ &+ \phi'_{\delta\varepsilon}\left(\widehat{X}^H_{\underline{t}^H}\right) c_\Delta\left(\widehat{X}^H_{\underline{t}^H}\right)\left(Z_t - Z_{\underline{t}^H}\right) + \frac{3\delta}{\varepsilon \log \delta}\Big[b^2\left(\widehat{X}^H_{\underline{t}^H}\right)(t - \underline{t}^H)^2 \\ &+ \sigma^2_\Delta\left(\widehat{X}^H_{\underline{t}^H}\right)\left(W_t - W_{\underline{t}^H}\right)^2 + c^2_\Delta\left(\widehat{X}^H_{\underline{t}^H}\right)\left(Z_t - Z_{\underline{t}^H}\right)^2\Big].\end{aligned}$$

Using (2.24), we get

$$b^2(\widehat{X}^H_{\underline{t}^H})(t - \underline{t}^H) \leq b^2\left(\widehat{X}^H_{\underline{t}^H}\right) h\left(\widehat{X}^H_{\underline{t}^H}\right)\Delta \leq \Delta.$$

Using **YW3**, **T2** and the fact that $|x|\phi'_{\delta\varepsilon}(|x|) \leq \phi_{\delta\varepsilon}(x) + \varepsilon$, we get

$$\begin{aligned}
&\phi'_{\delta\varepsilon}\left(\widehat{X}^H_{\underline{t}^H}\right)\left[b\left(\widehat{X}^H_{\underline{t}^H}\right) - b(0)\right](t - \underline{t}^H)\\
&= \frac{\phi'_{\delta\varepsilon}\left(|\widehat{X}^H_{\underline{t}^H}|\right)}{|\widehat{X}^H_{\underline{t}^H}|}\widehat{X}^H_{\underline{t}^H}\left[b\left(\widehat{X}^H_{\underline{t}^H}\right) - b(0)\right](t - \underline{t}^H)\\
&\leq L\left|\widehat{X}^H_{\underline{t}^H}\right|\phi'_{\delta\varepsilon}\left(|\widehat{X}^H_{\underline{t}^H}|\right)(t - \underline{t}^H)\\
&\leq L\left(\phi_{\delta\varepsilon}\left(\widehat{X}^H_{\underline{t}^H}\right) + \varepsilon\right)(t - \underline{t}^H).
\end{aligned}$$

Now, using **T4** and the fact that $dW_t^2 = 2W_t dW_t + dt$, we have

$$\sigma^2_\Delta\left(\widehat{X}^H_{\underline{t}^H}\right)\left(W_t - W_{\underline{t}^H}\right)^2 \leq \frac{L^2}{\Delta}\left(2\int_{\underline{t}^H}^t \left(W_s - W_{\underline{s}^H}\right)dW_s + (t - \underline{t}^H)\right).$$

Moreover, the Itô's formula yields

$$\left(Z_t - Z_{\underline{t}^H}\right)^2 = (t - \underline{t}^H)\int_{\mathbb{R}_0} z^2\nu(dz) + \int_{\underline{t}^H}^t\int_{\mathbb{R}_0}\left[z^2 + 2z\left(Z_{s-} - Z_{\underline{t}^H}\right)\right]\widetilde{N}(ds, dz).$$

This, together with **T4**, gives

$$\begin{aligned}
c^2_\Delta\left(\widehat{X}^H_{\underline{t}^H}\right)\left(Z_t - Z_{\underline{t}^H}\right)^2 \leq& \frac{L^2\left(t - \underline{t}^H\right)}{\Delta}\int_{\mathbb{R}_0} z^2\nu(dz)\\
&+ \frac{L^2}{\Delta}\int_{\underline{t}^H}^t\int_{\mathbb{R}_0}\left[z^2 + 2z\left(Z_{s-} - Z_{\underline{s}^H}\right)\right]\widetilde{N}(ds, dz).
\end{aligned}$$

Consequently, we have shown that

$$\begin{aligned}
&\phi_{\delta\varepsilon}\left(\widehat{X}^H_t\right) \leq \left[1 + L\left(t - \underline{t}^H\right)\right]\phi_{\delta\varepsilon}\left(\widehat{X}^H_{\underline{t}^H}\right)\\
&\quad + \left(L\varepsilon + |b(0)| + \frac{3\delta\Delta}{\varepsilon\log\delta} + \frac{3\delta L^2}{\Delta\varepsilon\log\delta} + \frac{3\delta L^2}{\Delta\varepsilon\log\delta}\int_{\mathbb{R}_0} z^2\nu(dz)\right)\left(t - \underline{t}^H\right)\\
&\quad + \int_{\underline{t}^H}^t\left[\phi'_{\delta e}\left(\widehat{X}^H_{\underline{s}^H}\right)\sigma_\Delta\left(\widehat{X}^H_{\underline{s}^H}\right) + \frac{6\delta L^2}{\Delta\varepsilon\log\delta}\left(W_s - W_{\underline{s}^H}\right)\right]dW_s\\
&\quad + \int_{\underline{t}^H}^t\int_{\mathbb{R}_0}\left[\phi'_{\delta\varepsilon}\left(\widehat{X}^H_{\underline{s}^H}\right)c_\Delta\left(\widehat{X}^H_{\underline{s}^H}\right)z + \frac{3\delta L^2}{\Delta\varepsilon\log\delta}\left(z^2 + 2z\left(Z_{s-} - Z_{\underline{s}^H}\right)\right)\right]\widetilde{N}(ds, dz).
\end{aligned} \tag{2.26}$$

Note that $e^{-Lt}(1 + L(t - \underline{t}^H)) \leq e^{-L\underline{t}^H}$. Then, multiplying by e^{-Lt} in both sides of (2.26), we have

$$e^{-Lt}\phi_{\delta\varepsilon}\left(\widehat{X}^H_t\right) \leq e^{-L\underline{t}^H}\phi_{\delta\varepsilon}\left(\widehat{X}^H_{\underline{t}^H}\right) + C(b(0), L, \Delta, \varepsilon, \delta)\int_{\underline{t}^H}^t e^{-Ls}ds$$

$$+e^{-Lt}\int_{\underline{t}^H}^{t}\left[\phi'_{\delta e}\left(\widehat{X}^H_{\underline{s}^H}\right)\sigma_\Delta\left(\widehat{X}^H_{\underline{s}^H}\right)+\frac{6\delta L^2}{\Delta\varepsilon\log\delta}\left(W_s-W_{\underline{s}^H}\right)\right]dW_s$$

$$+e^{-Lt}\int_{\underline{t}^H}^{t}\int_{\mathbb{R}_0}\left[\phi'_{\delta\varepsilon}\left(\widehat{X}^H_{\underline{s}^H}\right)c_\Delta\left(\widehat{X}^H_{\underline{s}^H}\right)z\right.$$

$$\left.+\frac{3\delta L^2}{\Delta\varepsilon\log\delta}\left(z^2+2z\left(Z_{s-}-Z_{\underline{s}^H}\right)\right)\right]\widetilde{N}(ds,dz). \tag{2.27}$$

Now, from (2.27), we get

$$e^{-Lt^H_{k+1}}\phi_{\delta\varepsilon}\left(\widehat{X}^H_{t^H_{k+1}}\right)$$

$$\leq e^{-Lt^H_k}\phi_{\delta\varepsilon}\left(\widehat{X}^H_{t^H_k}\right)+C(b(0),L,\Delta,\varepsilon,\delta)\int_{t^H_k}^{t^H_{k+1}}e^{-Ls}ds$$

$$+e^{-Lt^H_{k+1}}\int_{t^H_k}^{t^H_{k+1}}\left[\phi'_{\delta e}\left(\widehat{X}^H_{\underline{s}^H}\right)\sigma_\Delta\left(\widehat{X}^H_{\underline{s}^H}\right)+\frac{6\delta L^2}{\Delta\varepsilon\log\delta}\left(W_s-W_{\underline{s}^H}\right)\right]dW_s$$

$$+e^{-Lt^H_{k+1}}\int_{t^H_k}^{t^H_{k+1}}\int_{\mathbb{R}_0}\left[\phi'_{\delta\varepsilon}\left(\widehat{X}^H_{\underline{s}^H}\right)c_\Delta\left(\widehat{X}^H_{\underline{s}^H}\right)z\right.$$

$$\left.+\frac{3\delta L^2}{\Delta\varepsilon\log\delta}\left(z^2+2z\left(Z_{s-}-Z_{\underline{s}^H}\right)\right)\right]\widetilde{N}(ds,dz). \tag{2.28}$$

Summing (2.28) over multiple timesteps and adding (2.27), we obtain

$$e^{-Lt}\phi_{\delta\varepsilon}\left(\widehat{X}^H_t\right)$$

$$\leq \phi_{\delta\varepsilon}(x_0)+C(b(0),L,\Delta,\varepsilon,\delta)\int_0^t e^{-Ls}ds$$

$$+\int_0^t e^{-L\left(\underline{s}^H+h\left(\widehat{X}^H_{\underline{s}^H}\right)\right)}\left[\phi'_{\delta e}\left(\widehat{X}^H_{\underline{s}H}\right)\sigma_\Delta\left(\widehat{X}^H_{\underline{s}^H}\right)+\frac{6\delta L^2}{\Delta\varepsilon\log\delta}\left(W_s-W_{\underline{s}^H}\right)\right]dW_s$$

$$+\int_0^t\int_{\mathbb{R}_0}e^{-L\left(\underline{s}^H+h\left(\widehat{X}^H_{\underline{s}^H}\right)\right)}\left[\phi'_{\delta\varepsilon}\left(\widehat{X}^H_{\underline{s}^H}\right)c_\Delta\left(\widehat{X}^H_{\underline{s}^H}\right)z\right.$$

$$\left.+\frac{3\delta L^2}{\Delta\varepsilon\log\delta}\left(z^2+2z\left(Z_{s-}-Z_{\underline{s}^H}\right)\right)\right]\widetilde{N}(ds,dz)$$

$$+\left(e^{-Lt}-e^{-L\left(\underline{t}^H+h\left(\widehat{X}^H_{\underline{t}^H}\right)\right)}\right)\int_{\underline{t}^H}^{t}\left[\phi'_{\delta\varepsilon}\left(\widehat{X}^H_{\underline{s}^H}\right)\sigma_\Delta\left(\widehat{X}^H_{\underline{s}^H}\right)\right.$$

$$\left.+\frac{6\delta L^2}{\Delta\varepsilon\log\delta}\left(W_s-W_{\underline{s}^H}\right)\right]dW_s$$

$$+\left(e^{-Lt}-e^{-L\left(\underline{t}^H+h\left(\widehat{X}^H_{\underline{t}^H}\right)\right)}\right)\int_{\underline{t}^H}^{t}\int_{\mathbb{R}_0}\left[\phi'_{\delta\varepsilon}\left(\widehat{X}^H_{\underline{s}^H}\right)c_\Delta\left(\widehat{X}^H_{\underline{s}^H}\right)z\right.$$

$$\left.+\frac{3\delta L^2}{\Delta\varepsilon\log\delta}\left(z^2+2z\left(Z_{s-}-Z^H_{\underline{s}}\right)\right)\right]\widetilde{N}(ds,dz)$$

$$\leq \phi_{\delta\varepsilon}(x_0) + C(b(0), L, \Delta, \varepsilon, \delta) \int_0^t e^{-Ls} ds$$

$$+ \int_0^t e^{-L\left(\underline{s}^H + h\left(\widehat{X}^H_{\underline{s}^H}\right)\right)} \left[\phi'_{\delta e}\left(\widehat{X}^H_{\underline{s}H}\right) \sigma_\Delta\left(\widehat{X}^H_{\underline{s}^H}\right) + \frac{6\delta L^2}{\Delta\varepsilon \log\delta}\left(W_s - W_{\underline{s}^H}\right)\right] dW_s$$

$$+ \int_0^t \int_{\mathbb{R}_0} e^{-L\left(\underline{s}^H + h\left(\widehat{X}^H_{\underline{s}^H}\right)\right)} \left[\phi'_{\delta\varepsilon}\left(\widehat{X}^H_{\underline{s}^H}\right) c_\Delta\left(\widehat{X}^H_{\underline{s}^H}\right) z\right.$$

$$\left. + \frac{3\delta L^2}{\Delta\varepsilon \log\delta}\left(z^2 + 2z\left(Z_{s-} - Z_{\underline{s}^H}\right)\right)\right] \widetilde{N}(ds, dz)$$

$$+ L\Delta\left(\frac{2L}{\sqrt{\Delta}} \sup_{0\leq s\leq t} |W_s| + \frac{12\delta L^2}{\Delta\varepsilon\log\delta} \sup_{0\leq s\leq t} |W_s|^2\right)$$

$$+ L\Delta\left(\frac{2L}{\sqrt{\Delta}} \sup_{0\leq s\leq t} |Z_s| + \frac{12\delta L^2}{\Delta\varepsilon\log\delta} \sup_{0\leq s\leq t} |Z_s|^2\right).$$

Now, using Lemma 1.2.25, we have

$$\mathbb{E}\left[\sup_{0\leq s\leq t} |Z_s|\right] \leq C\sqrt{t}\left(\int_{\mathbb{R}_0} z^2\nu(dz)\right)^{\frac{1}{2}}; \quad \mathbb{E}\left[\sup_{0\leq s\leq t} Z_s^2\right] \leq Ct\int_{\mathbb{R}_0} z^2\nu(dz).$$

Hence, for any the stopping time $\tau \leq t$, it follows from the modulus of continuity of W and the Condition **B6** that there exists a positive constant $C = C(x_0, \varepsilon, \delta, \Delta, L, b(0), t)$, which does not depend on H such that

$$\mathbb{E}\left[e^{-L\tau}\left|\widehat{X}^H_\tau\right|\right] \leq C(x_0, \varepsilon, \delta, \Delta, L, b(0), t).$$

From this point forward, by repeating the argument in the proof of Proposition 2.1 in [44] we obtain the desired result. □

Under the assumptions of Proposition 2.4.1, the nearest time point before t is defined by $\underline{t} := \max\{t_n : t_n \leq t\}$, and the number of time steps approximation up to time t is defined by $N_t := \max\{n : t_n \leq t\}$. Observe that $\underline{t}$ is a stopping time. Then, the standard continuous interpolant is defined by

$$\widehat{X}_t = \widehat{X}_{\underline{t}} + b\left(\widehat{X}_{\underline{t}}\right)(t - \underline{t}) + \sigma_\Delta\left(\widehat{X}_{\underline{t}}\right)\left(W_t - W_{\underline{t}}\right) + c_\Delta\left(\widehat{X}_{\underline{t}-}\right)\left(Z_t - Z_{\underline{t}}\right). \quad (2.29)$$

Hence, $\widehat{X} = (\widehat{X}_t)_{t\geq 0}$ is the solution of the following SDE

$$d\widehat{X}_t = b(\widehat{X}_{\underline{t}})dt + \sigma_\Delta\left(\widehat{X}_{\underline{t}}\right) dW_t + c_\Delta\left(\widehat{X}_{\underline{t}-}\right) dZ_t, \qquad \widehat{X}_0 = x_0. \quad (2.30)$$

2.5 Moments of the approximate solution

In Section 2.3, we established the boundedness of the moments of the exact solution in L^p (for $p \in (0, p_0]$), over both finite and infinite time horizons. In this section, we will also establish similar properties for the moments of the approximate solution.

Firstly, we require a crucial estimate for the moment of $\widehat{X}_t$.

Lemma 2.5.1. *Assume that Conditions **T1**–**T4** and **B6** hold. Then for any* $p \in [1; 2p_0]$ *and* $T > 0$, *there exists a positive constant* $C(p, L, T, x_0, \Delta)$ *such that*

$$\mathbb{E}\left[\sup_{0\leq t\leq T} |\widehat{X}_t|^p\right] \leq C(p, L, T, x_0, \Delta).$$

Proof. Using the property **YW3** and applying Itô's formula for $e^{-Lt}\phi_{\delta\varepsilon}(\widehat{X}_t)$, we get

$$\begin{aligned}
& e^{-Lt}|\widehat{X}_t| \\
&\leq e^{-Lt}\varepsilon + e^{-Lt}\phi_{\delta\varepsilon}(\widehat{X}_t) \\
&\leq \varepsilon + \phi_{\delta\varepsilon}(x_0) + \int_0^t e^{-Ls}\left[-L\phi_{\delta\varepsilon}(\widehat{X}_s) + \phi'_{\delta\varepsilon}(\widehat{X}_s)b(\widehat{X}_{\underline{s}}) + \frac{1}{2}\phi''_{\delta\varepsilon}(\widehat{X}_s)\sigma^2_\Delta(\widehat{X}_{\underline{s}})\right] ds \\
&\quad + \int_0^t e^{-Ls}\phi'_{\delta\varepsilon}(\widehat{X}_s)\sigma_\Delta(\widehat{X}_{\underline{s}})dW_s \\
&\quad + \int_0^t\int_{\mathbb{R}_0} e^{-Ls}\left[\phi_{\delta\varepsilon}\left(\widehat{X}_s + c_\Delta(\widehat{X}_{\underline{s}})z\right) - \phi_{\delta\varepsilon}\left(\widehat{X}_s\right) - \phi'_{\delta\varepsilon}\left(\widehat{X}_s\right)c_\Delta\left(\widehat{X}_{\underline{s}}\right)z\right]\nu(dz)ds \\
&\quad + \int_0^t\int_{\mathbb{R}_0} e^{-Ls}\left[\phi_{\delta\varepsilon}\left(\widehat{X}_{s-} + c_\Delta\left(\widehat{X}_{\underline{s}}\right)z\right) - \phi_{\delta\varepsilon}\left(\widehat{X}_{s-}\right)\right]\widetilde{N}(ds, dz).
\end{aligned}$$

Now, applying Taylor's expansion for $\phi'_{\delta\varepsilon}$ and using **T2**, **YW2**, **YW5** and equation (2.29), there exists an $(\mathcal{F}_s)$-adapted process $\xi = (\xi_s)$ such that

$$\begin{aligned}
\phi'_{\delta\varepsilon}(\widehat{X}_s)b(\widehat{X}_{\underline{s}}) &= \left(\phi'_{\delta\varepsilon}\left(\widehat{X}_{\underline{s}}\right) + \phi''_{\delta\varepsilon}(\xi_s)\left(\widehat{X}_s - \widehat{X}_{\underline{s}}\right)\right)b\left(\widehat{X}_{\underline{s}}\right) \\
&= \phi'_{\delta\varepsilon}\left(\widehat{X}_{\underline{s}}\right)b\left(\widehat{X}_{\underline{s}}\right) + \phi''_{\delta\varepsilon}(\xi_s)\left[b\left(\widehat{X}_{\underline{s}}\right)(s-\underline{s}) + \sigma_\Delta\left(\widehat{X}_{\underline{s}}\right)(W_s - W_{\underline{s}})\right. \\
&\quad \left.+ c_\Delta\left(\widehat{X}_{\underline{s}}\right)(Z_s - Z_{\underline{s}})\right]b\left(\widehat{X}_{\underline{s}}\right) \\
&= \phi'_{\delta\varepsilon}\left(\widehat{X}_{\underline{s}}\right)\left(b\left(\widehat{X}_{\underline{s}}\right) - b(0)\right) + \phi'_{\delta\varepsilon}\left(\widehat{X}_{\underline{s}}\right)b(0) \\
&\quad + \phi''_{\delta\varepsilon}(\xi_s)\left[b^2\left(\widehat{X}_{\underline{s}}\right)(s-\underline{s}) + b\left(\widehat{X}_{\underline{s}}\right)\sigma_\Delta\left(\widehat{X}_{\underline{s}}\right)(W_s - W_{\underline{s}})\right. \\
&\quad \left.+ b\left(\widehat{X}_{\underline{s}}\right)c_\Delta\left(\widehat{X}_{\underline{s}}\right)(Z_s - Z_{\underline{s}})\right] \\
&= \frac{\phi'_{\delta\varepsilon}\left(\left|\widehat{X}_{\underline{s}}\right|\right)}{\left|\widehat{X}_{\underline{s}}\right|}\widehat{X}_{\underline{s}}\left(b\left(\widehat{X}_{\underline{s}}\right) - b(0)\right) + \phi'_{\delta\varepsilon}\left(\widehat{X}_{\underline{s}}\right)b(0) \\
&\quad + \phi''_{\delta\varepsilon}(\xi_s)\left[b^2\left(\widehat{X}_{\underline{s}}\right)(s-\underline{s}) + b\left(\widehat{X}_{\underline{s}}\right)\sigma_\Delta\left(\widehat{X}_{\underline{s}}\right)(W_s - W_{\underline{s}})\right. \\
&\quad \left.+ b\left(\widehat{X}_{\underline{s}}\right)c_\Delta\left(\widehat{X}_{\underline{s}}\right)(Z_s - Z_{\underline{s}})\right]
\end{aligned}$$

$$\le L\left|\widehat{X}_{\underline{s}}\right| + |b(0)| + \frac{C\delta\Delta}{\varepsilon\log\delta} + \frac{2\delta}{\varepsilon\log\delta}\left|b\left(\widehat{X}_{\underline{s}}\right)\sigma_\Delta\left(\widehat{X}_{\underline{s}}\right)(W_s - W_{\underline{s}})\right|$$
$$+ \frac{2\delta}{\varepsilon\log\delta}\left|b\left(\widehat{X}_{\underline{s}}\right)c_\Delta\left(\widehat{X}_{\underline{s}}\right)(Z_s - Z_{\underline{s}})\right|.$$

Again, applying Taylor's expansion for $\phi_{\delta\varepsilon}$ and using **YW5**, **T4**, there exists an $(\mathcal{F}_s)$-adapted process $\theta = (\theta_s)$ such that

$$\begin{aligned}
&\phi_{\delta\varepsilon}\left(\widehat{X}_s + c_\Delta\left(\widehat{X}_{\underline{s}}\right)z\right) - \phi_{\delta\varepsilon}\left(\widehat{X}_s\right) - \phi'_{\delta\varepsilon}\left(\widehat{X}_s\right)c_\Delta\left(\widehat{X}_{\underline{s}}\right)z \\
&= \frac{1}{2}\phi''_{\delta\varepsilon}(\theta_s)\,c^2_\Delta\left(\widehat{X}_{\underline{s}}\right)z^2 \\
&\le \frac{\delta}{\varepsilon\log\delta}\frac{L^2}{\Delta}z^2.
\end{aligned}$$

Hence, we have shown that

$$\begin{aligned}
e^{-Lt}|\widehat{X}_t| \le\; & \varepsilon + \phi_{\delta\varepsilon}(x_0) + \int_0^t e^{-Ls}\left[-L\phi_{\delta\varepsilon}(\widehat{X}_s) + L\left|\widehat{X}_{\underline{s}}\right| + |b(0)|\right. \\
&+ \frac{C\delta\Delta}{\varepsilon\log\delta} + \frac{2\delta}{\varepsilon\log\delta}\left|b\left(\widehat{X}_{\underline{s}}\right)\sigma_\Delta\left(\widehat{X}_{\underline{s}}\right)(W_s - W_{\underline{s}})\right| \\
&\left.+\frac{2\delta}{\varepsilon\log\delta}\left|b\left(\widehat{X}_{\underline{s}}\right)c_\Delta\left(\widehat{X}_{\underline{s}}\right)(Z_s - Z_{\underline{s}})\right| + \frac{L\delta}{\Delta\varepsilon\log\delta}\right]ds \\
&+ \int_0^t e^{-Ls}\phi'_{\delta\varepsilon}(\widehat{X}_s)\sigma_\Delta(\widehat{X}_{\underline{s}})dW_s + \frac{L^2\delta}{\Delta\varepsilon\log\delta}\int_0^t e^{-Ls}ds\int_{\mathbb{R}_0} z^2\nu(dz) \\
&+ \int_0^t\int_{\mathbb{R}_0} e^{-Ls}\left[\phi_{\delta\varepsilon}\left(\widehat{X}_{s-} + c_\Delta\left(\widehat{X}_{\underline{s}}\right)z\right) - \phi_{\delta\varepsilon}\left(\widehat{X}_{s-}\right)\right]\widetilde{N}(ds,dz).
\end{aligned}$$

Next, using **YW3** and (2.29), we have

$$\begin{aligned}
-\phi_{\delta\varepsilon}(\widehat{X}_s) + \left|\widehat{X}_{\underline{s}}\right| &\le -|\widehat{X}_s| + \varepsilon + \left|\widehat{X}_{\underline{s}}\right| \\
&\le \varepsilon + |b(\widehat{X}_{\underline{s}})(s-\underline{s})| + |\sigma_\Delta(\widehat{X}_{\underline{s}})(W_s - W_{\underline{s}})| + |c_\Delta(\widehat{X}_{\underline{s}})(Z_s - Z_{\underline{s}})|.
\end{aligned}$$

Thus,

$$\begin{aligned}
&e^{-Lt}|\widehat{X}_t| \\
&\le (1+Lt)\varepsilon + \phi_{\delta\varepsilon}(x_0) \\
&+ \int_0^t Le^{-Ls}\left[|b(\widehat{X}_{\underline{s}})(s-\underline{s})| + |\sigma_\Delta(\widehat{X}_{\underline{s}})(W_s - W_{\underline{s}})| + |c_\Delta(\widehat{X}_{\underline{s}})(Z_s - Z_{\underline{s}})|\right]ds \\
&+ \int_0^t e^{-Ls}\left[|b(0)| + \frac{C\delta\Delta}{\varepsilon\log\delta} + \frac{2\delta}{\varepsilon\log\delta}\left|b\left(\widehat{X}_{\underline{s}}\right)\sigma_\Delta\left(\widehat{X}_{\underline{s}}\right)(W_s - W_{\underline{s}})\right|\right. \\
&\left.+\frac{2\delta}{\varepsilon\log\delta}\left|b\left(\widehat{X}_{\underline{s}}\right)c_\Delta\left(\widehat{X}_{\underline{s}}\right)(Z_s - Z_{\underline{s}})\right| + \frac{C\delta}{\Delta\varepsilon\log\delta}\right]ds
\end{aligned}$$

$$+\int_0^t e^{-Ls}\phi'_{\delta\varepsilon}(\widehat{X}_s)\sigma_\Delta(\widehat{X}_{\underline{s}})dW_s + \frac{CL^2\delta}{\Delta\varepsilon\log\delta}\int_0^t e^{-Ls}ds$$
$$+\int_0^t\int_{\mathbb{R}_0} e^{-Ls}\left[\phi_{\delta\varepsilon}\left(\widehat{X}_{s-}+c_\Delta\left(\widehat{X}_{\underline{s}}\right)z\right)-\phi_{\delta\varepsilon}\left(\widehat{X}_{s-}\right)\right]\widetilde{N}(ds,dz). \tag{2.31}$$

Using (2.25), for any $p\in[1;2p_0]$, there exists a constant $C(p)>0$ such that

$$\begin{aligned}\mathbb{E}\left[\left|b\left(\widehat{X}_{\underline{s}}\right)\sigma_\Delta\left(\widehat{X}_{\underline{s}}\right)(W_s-W_{\underline{s}})\right|^p\right] &= \mathbb{E}\left[\left|b\left(\widehat{X}_{\underline{s}}\right)\sigma_\Delta\left(\widehat{X}_{\underline{s}}\right)\right|^p\mathbb{E}\left[|W_s-W_{\underline{s}}|^p\middle|\mathcal{F}_{\underline{s}}\right]\right]\\ &\le C(p)\mathbb{E}\left[\left|b\left(\widehat{X}_{\underline{s}}\right)\sigma_\Delta\left(\widehat{X}_{\underline{s}}\right)\right|^p(s-\underline{s})^{p/2}\right]\\ &\le C(p)\mathbb{E}\left[\left|b\left(\widehat{X}_{\underline{s}}\right)\sigma_\Delta\left(\widehat{X}_{\underline{s}}\right)\right|^p\left|h(\widehat{X}_{\underline{s}})\right|^{p/2}\Delta^{p/2}\right].\end{aligned}$$

Next, using the Burkholder-Davis-Gundy's inequality with jumps and **B6**, for any $p\in[1;2p_0]$, there exists a constant $C(p)>0$ such that

$$\begin{aligned}&\mathbb{E}\left[\left|b\left(\widehat{X}_{\underline{s}}\right)c_\Delta\left(\widehat{X}_{\underline{s}}\right)(Z_s-Z_{\underline{s}})\right|^p\right]\\ &=\mathbb{E}\left[\left|b\left(\widehat{X}_{\underline{s}}\right)c_\Delta\left(\widehat{X}_{\underline{s}}\right)\right|^p\mathbb{E}\left[|Z_s-Z_{\underline{s}}|^p\middle|\mathcal{F}_{\underline{s}}\right]\right]\\ &\le C(p)\mathbb{E}\left[\left|b\left(\widehat{X}_{\underline{s}}\right)c_\Delta\left(\widehat{X}_{\underline{s}}\right)\right|^p\left(\int_{\underline{s}}^s\int_{\mathbb{R}_0}|z|^{2\vee p}\nu(dz)ds\right)^{1\wedge p/2}\right]\\ &=C(p)\left(\int_{\mathbb{R}_0}|z|^{2\vee p}\nu(dz)\right)^{1\wedge p/2}\mathbb{E}\left[\left|b\left(\widehat{X}_{\underline{s}}\right)c_\Delta\left(\widehat{X}_{\underline{s}}\right)\right|^p(s-\underline{s})^{1\wedge p/2}\right]\\ &\le C(p)\mathbb{E}\left[\left|b\left(\widehat{X}_{\underline{s}}\right)c_\Delta\left(\widehat{X}_{\underline{s}}\right)\right|^p\left|h(\widehat{X}_{\underline{s}})\right|^{1\wedge p/2}\Delta^{1\wedge p/2}\right].\end{aligned}$$

By (2.24) and Conditions **T3, T4**, we have

$$\max\left\{\mathbb{E}\left[\left|b\left(\widehat{X}_{\underline{s}}\right)\sigma_\Delta\left(\widehat{X}_{\underline{s}}\right)(W_s-W_{\underline{s}})\right|^p\right];\mathbb{E}\left[\left|b\left(\widehat{X}_{\underline{s}}\right)c_\Delta\left(\widehat{X}_{\underline{s}}\right)(Z_s-Z_{\underline{s}})\right|^p\right]\right\}$$
$$\le C(p,L,\Delta).$$

Therefore, by choosing $\varepsilon=1,\delta=2$ in (2.31), it follows from **T1–T4**, Hölder's inequality and Burkholder-Davis-Gundy's inequality that for any $T>0$, there exists a positive constant $C(p,L,T,x_0,\Delta)<\infty$ such that

$$\mathbb{E}\left[\sup_{0\le t\le T}|\widehat{X}_t|^p\right]\le C(p,L,T,x_0,\Delta).$$

This finishes the proof. □

Next, utilizing Lemma 2.5.1, we establish the following bound on the moments of the tamed-adaptive Euler-Maruyama approximation. Our proof technique remains consistent, relying on mathematical induction, the binomial theorem, and Hölder inequality, mirroring the approach taken in Proposition 2.3.1.

Theorem 2.5.2. *Assume that Conditions **T1**–**T4** and **B6** hold, and for some $p_0 \in [2;+\infty)$, there exist constants $\gamma \in \mathbb{R}$, $\eta \in [0,+\infty)$ such that for all $x \in \mathbb{R}$,*

$$xb(x) + \frac{p_0-1}{2}\sigma_\Delta^2(x) + \frac{c_\Delta^2(x)}{2L_0}\int_{\mathbb{R}_0} |z| \left((1+L_0|z|)^{p_0-1} - 1\right)\nu(dz) \leq \gamma x^2 + \eta. \quad (2.32)$$

Then, for any positive integer $k \leq p_0/2$, there exists a positive constant $C = C(x_0, k, \eta, \gamma, L, L_0, p_0)$ which does not depend neither on t nor on Δ such that

$$\mathbb{E}\left[|\widehat{X}_t|^{2k}\right] \vee \mathbb{E}\left[|\widehat{X}_{\underline{t}}|^{2k}\right] \leq \begin{cases} Ce^{2k\gamma t} & \text{if } \gamma > 0, \\ C(1+t)^k & \text{if } \gamma = 0, \\ C & \text{if } \gamma < 0. \end{cases} \quad (2.33)$$

Remark 2.5.3. Since $(1+L_0|z|)^x$ is an increasing function for $x \geq 1$, we deduce from condition (2.32) that for any $p \in [2, p_0]$ and $x \in \mathbb{R}$,

$$xb(x) + \frac{p-1}{2}\sigma_\Delta^2(x) + \frac{c_\Delta^2(x)}{2L_0}\int_{\mathbb{R}_0} |z| \left((1+L_0|z|)^{p-1} - 1\right)\nu(dz) \leq \gamma x^2 + \eta.$$

Proof of Theorem 2.5.2. Using Hölder's inequality, it is sufficient to show (2.33) for a positive integer k and $k \leq p_0/2$. We will use the induction method.
Firstly, for $k = 1$, applying Itô's formula to $e^{-2\gamma t}\widehat{X}_t^2$, we get

$$\begin{aligned} e^{-2\gamma t}\widehat{X}_t^2 &= x_0^2 + 2\int_0^t e^{-2\gamma s}\left(-\gamma \widehat{X}_s^2 + \widehat{X}_s b(\widehat{X}_{\underline{s}}) + \frac{1}{2}\sigma_\Delta^2(\widehat{X}_{\underline{s}}) + \frac{1}{2}c_\Delta^2(\widehat{X}_{\underline{s}})\int_{\mathbb{R}_0} z^2\nu(dz)\right)ds \\ &\quad + 2\int_0^t e^{-2\gamma s}\widehat{X}_s\sigma_\Delta(\widehat{X}_{\underline{s}})dW_s \\ &\quad + \int_0^t\int_{\mathbb{R}_0} e^{-2\gamma s}\left[2\widehat{X}_{\underline{s}-}c_\Delta\left(\widehat{X}_{\underline{s}-}\right)z + c_\Delta^2\left(\widehat{X}_{\underline{s}-}\right)z^2\right]\widetilde{N}(ds,dz). \end{aligned} \quad (2.34)$$

It follows from (2.29) that

$$\begin{aligned} \widehat{X}_s^2 &= \widehat{X}_{\underline{s}}^2 + 2\widehat{X}_{\underline{s}}\left(b\left(\widehat{X}_{\underline{s}}\right)(s-\underline{s}) + \sigma_\Delta\left(\widehat{X}_{\underline{s}}\right)(W_s - W_{\underline{s}}) + c_\Delta\left(\widehat{X}_{\underline{s}}\right)(Z_s - Z_{\underline{s}})\right) \\ &\quad + \left(b\left(\widehat{X}_{\underline{s}}\right)(s-\underline{s}) + \sigma_\Delta\left(\widehat{X}_{\underline{s}}\right)(W_s - W_{\underline{s}}) + c_\Delta\left(\widehat{X}_{\underline{s}}\right)(Z_s - Z_{\underline{s}})\right)^2. \end{aligned}$$

Using **T3**, **T4**, **B6**, (2.25), and (2.24),

$$\begin{aligned} &\max\left\{|\widehat{X}_{\underline{s}}b(\widehat{X}_{\underline{s}})(s-\underline{s})|;\ b^2(\widehat{X}_{\underline{s}})(s-\underline{s})^2\right\} \leq C\Delta. \\ &\max\left\{\mathbb{E}\left[\sigma_\Delta^2(\widehat{X}_{\underline{s}})(W_s - W_{\underline{s}})^2\middle|\mathcal{F}_{\underline{s}}\right];\ \mathbb{E}\left[c_\Delta^2\left(\widehat{X}_{\underline{s}}\right)(Z_s - Z_{\underline{s}})^2\middle|\mathcal{F}_{\underline{s}}\right]\right\} \leq C\Delta. \end{aligned} \quad (2.35)$$

Therefore,

$$\mathbb{E}\left[-\gamma \widehat{X}_s^2\right] \le \mathbb{E}\left[-\gamma \widehat{X}_{\underline{s}}^2\right] + C|\gamma|\Delta. \tag{2.36}$$

A similar argument yields

$$\mathbb{E}\left[\widehat{X}_s b(\widehat{X}_{\underline{s}})\right] \le \mathbb{E}\left[\widehat{X}_{\underline{s}} b(\widehat{X}_{\underline{s}})\right] + C\Delta. \tag{2.37}$$

By Lemma 2.5.1, the expectation of the stochastic integrals in (2.34) is zero. It then follows from (2.32), (2.34), (2.36), (2.37) that

$$\begin{aligned}
\mathbb{E}\left[e^{-2\gamma t}\widehat{X}_t^2\right] &\le x_0^2 + 2\int_0^t e^{-2\gamma s}\Bigg(\mathbb{E}\Bigg[-\gamma \widehat{X}_{\underline{s}}^2 + \widehat{X}_{\underline{s}} b(\widehat{X}_{\underline{s}}) + \frac{1}{2}\sigma_\Delta^2(\widehat{X}_{\underline{s}}) \\
&\quad + \frac{1}{2}c_\Delta^2(\widehat{X}_{\underline{s}})\int_{\mathbb{R}_0} z^2 \nu(dz)\Bigg] + C\Delta\Bigg) ds \\
&\le x_0^2 + C(\eta+1)\int_0^t e^{-2\gamma s} ds.
\end{aligned} \tag{2.38}$$

Using the fact that

$$\widehat{X}_{\underline{t}} = \widehat{X}_t - b\left(\widehat{X}_{\underline{t}}\right)(t-\underline{t}) - \sigma_\Delta\left(\widehat{X}_{\underline{t}}\right)\left(W_t - W_{\underline{t}}\right) - c_\Delta\left(\widehat{X}_{\underline{t}-}\right)\left(Z_t - Z_{\underline{t}}\right)$$

together with (2.35), we get the following estimate for any $p > 1$

$$\begin{aligned}
\mathbb{E}\left[\left|\widehat{X}_{\underline{t}}\right|^p\right] &\le 4^{p-1}\Bigg(\mathbb{E}\left[\left|\widehat{X}_t\right|^p\right] + \mathbb{E}\left[\left|b(\widehat{X}_{\underline{t}})(t-\underline{t})\right|^p\right] + \mathbb{E}\left[\left|\sigma_\Delta(\widehat{X}_{\underline{t}})(W_t - W_{\underline{t}})\right|^p\right] \\
&\quad + \mathbb{E}\left[\left|c_\Delta(\widehat{X}_{\underline{t}})(Z_t - Z_{\underline{t}})\right|^p\right]\Bigg) \\
&\le 4^{p-1}\left(\mathbb{E}\left[\left|\widehat{X}_t\right|^p\right] + C\Delta^p + C\Delta^{p/2} + C\Delta^{1\wedge p/2}\right).
\end{aligned} \tag{2.39}$$

It follows from (2.38) and (2.39) that (2.33) holds for $k = 1$.

Secondly, assume that (2.33) holds for any $k \le k_0 \le [p_0/2] - 1$, we will show that (2.33) still holds for $k = k_0 + 1$. Here we use the notation $[p_0/2]$ for the integer part of $p_0/2$.

By applying Itô's formula for $e^{-p\gamma t}\widehat{X}_t^p$ with $p = 2(k_0+1)$ being an even integer, we have

$$\begin{aligned}
e^{-p\gamma t}\left|\widehat{X}_t\right|^p &= x_0^p + \int_0^t e^{-p\gamma s}\Bigg[-p\gamma \widehat{X}_s^p + p\widehat{X}_s^{p-1} b(\widehat{X}_{\underline{s}}) + \frac{p(p-1)}{2}\widehat{X}_s^{p-2}\sigma_\Delta^2(\widehat{X}_{\underline{s}}) \\
&\quad + \int_{\mathbb{R}_0}\left(\left(\widehat{X}_s + c_\Delta(\widehat{X}_{\underline{s}})z\right)^p - \widehat{X}_s^p - p\widehat{X}_s^{p-1}c_\Delta(\widehat{X}_{\underline{s}})z\right)\nu(dz)\Bigg] ds \\
&\quad + p\int_0^t e^{-p\gamma s}\widehat{X}_s^{p-2}\widehat{X}_s\sigma_\Delta(\widehat{X}_{\underline{s}}) dW_s
\end{aligned}$$

$$+\int_0^t\int_{\mathbb{R}_0} e^{-p\gamma s}\left(\left(\widehat{X}_{s-}+c_\Delta(\widehat{X}_{\underline{s}-})z\right)^p-\widehat{X}_{s-}^p\right)\widetilde{N}(ds,dz). \tag{2.40}$$

It follows from (2.29) and the binomial theorem that for any positive integer q,

$$\widehat{X}_s^q=\sum_{\substack{0\le i,j,r,v\le q\\ i+j+r+v=q}}\frac{q!}{i!j!r!v!}\left(\widehat{X}_{\underline{s}}\right)^i\left(b(\widehat{X}_{\underline{s}})(s-\underline{s})\right)^j\left(\sigma_\Delta(\widehat{X}_{\underline{s}})(W_s-W_{\underline{s}})\right)^r$$
$$\times\left(c_\Delta(\widehat{X}_{\underline{s}})(Z_s-Z_{\underline{s}})\right)^v. \tag{2.41}$$

Using (2.25), the independence between W and Z, Burkholder-Davis-Gundy's inequality and **B6**, we have

$$\begin{aligned}&\mathbb{E}\left[-\gamma\widehat{X}_s^p\big|\mathcal{F}_{\underline{s}}\right]\\ &\le-\gamma\widehat{X}_{\underline{s}}^p-p\gamma\widehat{X}_{\underline{s}}b(\widehat{X}_{\underline{s}})(s-\underline{s})\widehat{X}_{\underline{s}}^{p-2}\\ &+\sum_{\substack{0\le i\le p-2\\ i+j+2r=p}}\frac{C_p|\gamma|p!}{i!j!r!}\left|\widehat{X}_{\underline{s}}\right|^i\left|b(\widehat{X}_{\underline{s}})(s-\underline{s})\right|^j\left|\sigma_\Delta^2(\widehat{X}_{\underline{s}})(s-\underline{s})\right|^r\\ &+\sum_{\substack{0\le i\le p-2,v\ge2\\ i+j+2r+v=p}}\frac{C_p|\gamma|p!}{i!j!r!v!}\left|\widehat{X}_{\underline{s}}\right|^i\left|b(\widehat{X}_{\underline{s}})(s-\underline{s})\right|^j\left|\sigma_\Delta^2(\widehat{X}_{\underline{s}})(s-\underline{s})\right|^r\left|c_\Delta^v(\widehat{X}_{\underline{s}})(s-\underline{s})\right|.\end{aligned}$$

Again, using **T3**, **T4** and (2.24), we get

$$\mathbb{E}\left[-\gamma\widehat{X}_s^p\big|\mathcal{F}_{\underline{s}}\right]\le-\gamma\widehat{X}_{\underline{s}}^p+C_p|\gamma|\sum_{i=0}^{p-2}|\widehat{X}_{\underline{s}}|^i. \tag{2.42}$$

Choosing $q=p-1$ and $q=p-2$ in (2.41) and using the same argument, we get

$$\mathbb{E}\left[\widehat{X}_s^{p-1}b(\widehat{X}_{\underline{s}})\big|\mathcal{F}_{\underline{s}}\right]\le\widehat{X}_{\underline{s}}^{p-1}b(\widehat{X}_{\underline{s}})+C_p\sum_{i=0}^{p-2}|\widehat{X}_{\underline{s}}|^i \tag{2.43}$$

and

$$\mathbb{E}\left[\widehat{X}_s^{p-2}\sigma_\Delta^2(\widehat{X}_{\underline{s}})\big|\mathcal{F}_{\underline{s}}\right]\le\widehat{X}_{\underline{s}}^{p-2}\sigma_\Delta^2(\widehat{X}_{\underline{s}})+C_p\sum_{i=0}^{p-2}|\widehat{X}_{\underline{s}}|^i. \tag{2.44}$$

Now, using the binomial theorem, we have

$$\left(\widehat{X}_s+c_\Delta\left(\widehat{X}_{\underline{s}}\right)z\right)^p-\widehat{X}_s^p-p\widehat{X}_s^{p-1}c_\Delta\left(\widehat{X}_{\underline{s}}\right)z=\sum_{i=2}^p\binom{p}{i}\widehat{X}_s^{p-i}c_\Delta^i\left(\widehat{X}_{\underline{s}}\right)z^i. \tag{2.45}$$

Then, applying (2.41) to $q=p-j$ with $j\in\{2,\dots,p\}$, we obtain that, for $4\le i\le p$,

$$\mathbb{E}\left[\widehat{X}_s^{p-i}c_\Delta^i\left(\widehat{X}_{\underline{s}}\right)|\mathcal{F}_{\underline{s}}\right]\le\widehat{X}_{\underline{s}}^{p-i}c_\Delta^i\left(\widehat{X}_{\underline{s}}\right)+(p-i)\widehat{X}_{\underline{s}}b\left(\widehat{X}_{\underline{s}}\right)c_\Delta^i\left(\widehat{X}_{\underline{s}}\right)(s-\underline{s})\widehat{X}_{\underline{s}}^{p-i-2}$$

$$
\begin{aligned}
&+\mathbb{E}\left[Q_{p-2}\left(\widehat{X}_{\underline{s}}\right)|\mathcal{F}_{\underline{s}}\right] c_{\Delta}^{i}\left(\widehat{X}_{\underline{s}}\right) \\
&\leq \widehat{X}_{\underline{s}}^{p-i} c_{\Delta}^{i}\left(\widehat{X}_{\underline{s}}\right)+C_{p} Q_{p-2}\left(\widehat{X}_{\underline{s}}\right).
\end{aligned} \tag{2.46}
$$

Using (2.45), (2.46), and **T3**, **B1**, proceeding as in (2.6) and (2.7), we obtain that

$$
\begin{aligned}
&\mathbb{E}\left[\left(\widehat{X}_{s}+c_{\Delta}\left(\widehat{X}_{\underline{s}}\right) z\right)^{p}-\widehat{X}_{s}^{p}-p \widehat{X}_{s}^{p-1} c_{\Delta}\left(\widehat{X}_{\underline{s}}\right) z \middle| \mathcal{F}_{\underline{s}}\right] \\
&\leq \sum_{i=2}^{p}\binom{p}{i} \widehat{X}_{\underline{s}}^{p-i} c_{\Delta}^{i}\left(\widehat{X}_{\underline{s}}\right) z^{i}+C Q_{p-2}\left(\left|\widehat{X}_{\underline{s}}\right|, z\right) \\
&\leq c_{\Delta}^{2}\left(\widehat{X}_{\underline{s}}\right)|\widehat{X}_{\underline{s}}|^{p-2} \frac{p}{2 L_{0}}|z|\left((1+L_{0}|z|)^{p-1}-1\right) \\
&\quad+c_{\Delta}^{2}\left(\widehat{X}_{\underline{s}}\right) \sum_{i=3}^{p}\binom{p}{i} L_{0}^{i-2}\left(\frac{i-2}{2}|\widehat{X}_{\underline{s}}|^{p-4}+\sum_{j=2}^{i-2}\binom{i-2}{j}|\widehat{X}_{\underline{s}}|^{p-2-j}\right)|z|^{i} \\
&\quad+C Q_{p-2}\left(\left|\widehat{X}_{\underline{s}}\right|, z\right).
\end{aligned} \tag{2.47}
$$

Consequently, from (2.32), (2.42),(2.43), (2.44), (2.47), **B6** and Remark 2.5.3, and $c_{\Delta}^{2}(x) \leq c^{2}(x) \leq 2 L_{0}^{2}(1+x^{2})$ for any $x \in \mathbb{R}$, we obtain that

$$
\begin{aligned}
&\mathbb{E}\left[-p \gamma \widehat{X}_{s}^{p}+p \widehat{X}_{s}^{p-1} b(\widehat{X}_{\underline{s}})+\frac{p(p-1)}{2} \widehat{X}_{s}^{p-2} \sigma_{\Delta}^{2}(\widehat{X}_{\underline{s}})\right. \\
&\left.\quad+\int_{\mathbb{R}_{0}}\left(\left(\widehat{X}_{s}+c_{\Delta}(\widehat{X}_{\underline{s}}) z\right)^{p}-\widehat{X}_{s}^{p}-p \widehat{X}_{s}^{p-1} c_{\Delta}(\widehat{X}_{\underline{s}}) z\right) \nu(d z) \middle| \mathcal{F}_{\underline{s}}\right] \\
&\leq p|\widehat{X}_{\underline{s}}|^{p-2}\Bigg(-\gamma \widehat{X}_{\underline{s}}^{2}+\widehat{X}_{\underline{s}} b\left(\widehat{X}_{\underline{s}}\right)+\frac{p-1}{2} \sigma_{\Delta}^{2}\left(\widehat{X}_{\underline{s}}\right) \\
&\quad+\frac{c_{\Delta}^{2}(\widehat{X}_{\underline{s}})}{2 L_{0}} \int_{\mathbb{R}_{0}}|z|\left((1+L_{0}|z|)^{p-1}-1\right) \nu(d z)\Bigg)+Q_{p-2}\left(\left|\widehat{X}_{\underline{s}}\right|\right) \\
&\leq p \eta|\widehat{X}_{\underline{s}}|^{p-2}+Q_{p-2}\left(\left|\widehat{X}_{\underline{s}}\right|\right).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&\mathbb{E}\left[-p \gamma \widehat{X}_{s}^{p}+p \widehat{X}_{s}^{p-1} b(\widehat{X}_{\underline{s}})+\frac{p(p-1)}{2} \widehat{X}_{s}^{p-2} \sigma_{\Delta}^{2}(\widehat{X}_{\underline{s}})\right. \\
&\left.\quad+\int_{\mathbb{R}_{0}}\left(\left(\widehat{X}_{s}+c_{\Delta}(\widehat{X}_{\underline{s}}) z\right)^{p}-\widehat{X}_{s}^{p}-p \widehat{X}_{s}^{p-1} c_{\Delta}(\widehat{X}_{\underline{s}}) z\right) \nu(d z)\right] \\
&\leq C(p, \eta, L_{0}) \sum_{i=0}^{p-2} \mathbb{E}\left[\left|\widehat{X}_{\underline{s}}\right|^{i}\right].
\end{aligned} \tag{2.48}
$$

Thanks to Lemma 2.5.1, the expectation of the stochastic integrals in (2.40) is equal to zero. Then, from the estimates (2.39), (2.40), (2.48) and the inductive assumption, we obtain that (2.33) holds for $k = k_0 + 1$, which implies the desired result. □

Remark 2.5.4. If $\gamma < 0$, then the approximated solution is stable in the sense that for any $0 \leq p \leq 2[p_0/2]$ there exists a positive constant C, which does not depend on Δ, such that

$$\sup_{t\geq 0} \mathbb{E}\left[|\widehat{X}_t|^p\right] \vee \mathbb{E}\left[|\widehat{X}_{\underline{t}}|^p\right] < C.$$

Remark 2.5.5. Suppose that all conditions of Theorem 2.5.2 hold, then the bound on the expectation of the number of time steps N_T required by a path approximation on $[0, T]$ for any $T > 0$ is given by

$$\mathbb{E}\left[N_T - 1\right] \leq \frac{C}{\Delta}, \tag{2.49}$$

where C is a positive constant that does not depend on Δ.

Following the argument in the proof of Lemma 2 in [18], we can obtain the estimate (2.49) as a consequence of Lemma 2.5.1 and Theorem 2.5.2.

Remark 2.5.6. It is straightforward to verify that under Conditions **B7–B9** and $\int_{\mathbb{R}_0} z^2\nu(dz) < \infty$, the following functions

$$c_\Delta(x) = \frac{c(x)}{1 + \Delta^{1/2}|c(x)|(1 + |b(x)|)}, \quad \sigma_\Delta(x) = \frac{\sigma(x)}{1 + \Delta^{1/2}|\sigma(x)|} \tag{2.50}$$

satisfy all conditions of Proposition 2.4.1.

2.6 Convergence of the tamed-adaptive Euler-Maruyama scheme

In this section, we will assess the strong convergence rate of the tamed-adaptive Euler-Maruyama scheme, across both finite and infinite time intervals. To achieve this, we first require the following uniformly-in-time bound for the difference between the two approximate solutions $\widehat{X}$ and $\widehat{X}_{\underline{t}}$.

Lemma 2.6.1. *Suppose that coefficients $b, c, \sigma, \sigma_\Delta, c_\Delta$ and the Lévy measure ν satisfy all conditions of Theorem 2.5.2 and $p \in (0; p_0]$, then there exists a positive constant $C_p = C(p, L)$ such that*

$$\sup_{t\geq 0} \mathbb{E}\left[|\widehat{X}_t - \widehat{X}_{\underline{t}}|^p\right] \leq C_p\Delta^{1\wedge p/2},$$

Proof. From (2.29), for any $p \geq 1$,

$$\begin{aligned}
&|\widehat{X}_t - \widehat{X}_{\underline{t}}|^p \\
&= \left| b(\widehat{X}_{\underline{t}})(t-\underline{t}) + \sigma_\Delta(\widehat{X}_{\underline{t}})(W_t - W_{\underline{t}}) + c_\Delta(\widehat{X}_{\underline{t}})(Z_t - Z_{\underline{t}}) \right|^p \\
&\leq 3^{p-1} \left[\left| b(\widehat{X}_{\underline{t}})(t-\underline{t}) \right|^p + \left| \sigma_\Delta(\widehat{X}_{\underline{t}})(W_t - W_{\underline{t}}) \right|^p + \left| c_\Delta(\widehat{X}_{\underline{t}})(Z_t - Z_{\underline{t}}) \right|^p \right] \\
&\leq 3^{p-1} \left[\left| b(\widehat{X}_{\underline{t}}) \right|^p \left| h(\widehat{X}_{\underline{t}}) \right|^p \Delta^p + \left| \sigma_\Delta(\widehat{X}_{\underline{t}}) \right|^p \left| W_t - W_{\underline{t}} \right|^p + \left| c_\Delta(\widehat{X}_{\underline{t}}) \right|^p \left| Z_t - Z_{\underline{t}} \right|^p \right].
\end{aligned}$$

By applying **T3** and (2.24), we have

$$\left| b(\widehat{X}_{\underline{t}}) h(\widehat{X}_{\underline{t}}) \right| \leq C; \ \sigma_\Delta^2(\widehat{X}_{\underline{t}}) h(\widehat{X}_{\underline{t}}) \leq C \text{ and } \left| c_\Delta^p(\widehat{X}_{\underline{t}}) h(\widehat{X}_{\underline{t}})^{1 \wedge p/2} \right| \leq C,$$

for some positive constant C. Consequently, using Burkholder-Davis-Gundy's inequality, **B6** and (2.25), we obtain the desired result. For $0 < p < 1$, it suffices to use Hölder's inequality. □

Finally, we assess the convergence rate of the tamed-adaptive Euler-Maruyama scheme in the following main theorem.

Theorem 2.6.2. *Assume that Conditions **B2, B6–B10** hold and $p_0 \geq \max\{4l; 2 + 4\alpha + 4m\}$. Assume that the functions $c, b, \sigma, c_\Delta, \sigma_\Delta$ and the Lévy measure ν satisfy all conditions of Theorem 2.5.2, and*

$$|c(x) - c_\Delta(x)| \leq L_5 \Delta^{1/2} c^2(x)(1 + |b(x)|), \quad |\sigma(x) - \sigma_\Delta(x)| \leq L_5 \Delta^{1/2} \sigma^2(x), \tag{2.51}$$

for all $x \in \mathbb{R}$ and some constant $L_5 > 0$.

Then, for any $T > 0$, there exists a positive constant $C_T = C(x_0, L, L_0, L_1, L_2, L_3, L_4, L_5, \gamma, \eta, T)$ such that

$$\sup_{0 \leq t \leq T} \mathbb{E}\left[|\widehat{X}_t - X_t| \right] \leq \begin{cases} C_T \Delta^\alpha & \text{if } 0 < \alpha \leq \dfrac{1}{2}, \\ \dfrac{C_T}{\log \frac{1}{\Delta}} & \text{if } \alpha = 0. \end{cases} \tag{2.52}$$

Moreover, let $\mu := \int_{\mathbb{R}_0} |z| \nu(dz)$ and assume that $L_1 + 2L_4\mu < 0$, $\gamma < 0$, then there exists a positive constant $C = C(x_0, L, L_0, L_1, L_2, L_3, L_4, L_5, \gamma, \eta)$ which does not depend on T such that

$$\sup_{t \geq 0} \mathbb{E}\left[|\widehat{X}_t - X_t| \right] \leq \begin{cases} C \Delta^\alpha & \text{if } 0 < \alpha \leq \dfrac{1}{2}, \\ \dfrac{C}{\log \frac{1}{\Delta}} & \text{if } \alpha = 0. \end{cases} \tag{2.53}$$

Remark 2.6.3. Note that if c satisfies Condition **B10** then it also satisfies Condition **B1** with $L_0 = \max\{L_4, |c(0)|\}$. Moreover, conditions **B8, B9, B10** imply conditions **B3, B4, B5**. Therefore, under Conditions **B2**, **B8–B10** and $\int_{\mathbb{R}_0} |z|\nu(dz) < \infty$, $\int_{\mathbb{R}_0} z^2\nu(dz) < \infty$, equation (2.1) has a unique strong solution.

Remark 2.6.4. It is straightforward to verify that under Conditions **B2, B6–B10**, the functions c_Δ and σ_Δ defined in (2.50) satisfy condition (2.51).

Proof of Theorem 2.6.2. Put $Y_t = X_t - \widehat{X}_t$. For any $\lambda \in \mathbb{R}$, applying the property **YW3** and Itô's formula for $e^{-\lambda t}\phi_{\delta\varepsilon}(Y_t)$, we get

$$\begin{aligned}
&e^{-\lambda t}|Y_t| \leq e^{-\lambda t}\varepsilon + e^{-\lambda t}\phi_{\delta\varepsilon}(Y_t)\\
&= e^{-\lambda t}\varepsilon + \int_0^t e^{-\lambda s}\Big[-\lambda\phi_{\delta\varepsilon}(Y_s) + \phi'_{\delta\varepsilon}(Y_s)\left(b(X_s) - b(\widehat{X}_{\underline{s}})\right)\\
&\quad+\frac{1}{2}\phi''_{\delta\varepsilon}(Y_s)\left|\sigma(X_s) - \sigma_\Delta(\widehat{X}_{\underline{s}})\right|^2\Big]\,ds + \int_0^t e^{-\lambda s}\phi'_{\delta\varepsilon}(Y_s)\left(\sigma(X_s) - \sigma_\Delta(\widehat{X}_{\underline{s}})\right)dW_s\\
&\quad+\int_0^t\int_{\mathbb{R}_0} e^{-\lambda s}\Big[\phi_{\delta\varepsilon}\left(Y_s + \left(c(X_s) - c_\Delta\left(\widehat{X}_{\underline{s}}\right)\right)z\right) - \phi_{\delta\varepsilon}\left(Y_s\right)\\
&\quad-\phi'_{\delta\varepsilon}\left(Y_s\right)\left(c\left(X_s\right) - c_\Delta\left(\widehat{X}_{\underline{s}}\right)\right)z\Big]\nu(dz)ds\\
&\quad+\int_0^t\int_{\mathbb{R}_0} e^{-\lambda s}\Big[\phi_{\delta\varepsilon}\left(Y_{s-} + \left(c\left(X_s\right) - c_\Delta\left(\widehat{X}_{\underline{s}}\right)\right)z\right) - \phi_{\delta\varepsilon}\left(Y_{s-}\right)\Big]\widetilde{N}\left(ds,dz\right).
\end{aligned} \tag{2.54}$$

Set

$$\begin{aligned}
J_1(s) &= \phi'_{\delta\varepsilon}(Y_s)\left(b(X_s) - b(\widehat{X}_{\underline{s}})\right),\\
J_2(s) &= \frac{1}{2}\phi''_{\delta\varepsilon}(Y_s)\left|\sigma(X_s) - \sigma_\Delta(\widehat{X}_{\underline{s}})\right|^2,\\
J_3(s) &= \phi_{\delta\varepsilon}\left(Y_s + \left(c(X_s) - c_\Delta\left(\widehat{X}_{\underline{s}}\right)\right)z\right) - \phi_{\delta\varepsilon}\left(Y_s\right)\\
&\qquad - \phi'_{\delta\varepsilon}\left(Y_s\right)\left(c\left(X_s\right) - c_\Delta\left(\widehat{X}_{\underline{s}}\right)\right)z.
\end{aligned}$$

Firstly, using properties **YW1, YW2**, Conditions **B7, B8** and Cauchy's inequality, we have

$$\begin{aligned}
J_1(s) &\leq \frac{\phi'_{\delta\varepsilon}(|Y_s|)}{|Y_s|}Y_s\left(b(X_s) - b(\widehat{X}_s)\right) + \left|\phi'_{\delta\varepsilon}(Y_s)\left(b(\widehat{X}_s) - b(\widehat{X}_{\underline{s}})\right)\right|\\
&\leq L_1\phi'_{\delta\varepsilon}(|Y_s|)|Y_s| + L_2\left(1 + |\widehat{X}_s|^l + |\widehat{X}_{\underline{s}}|^l\right)|\widehat{X}_s - \widehat{X}_{\underline{s}}|\\
&\leq L_1|Y_s| + \frac{3}{2}L_2\Delta^{1/2}\left(1 + |\widehat{X}_s|^{2l} + |\widehat{X}_{\underline{s}}|^{2l}\right) + \frac{1}{2}L_2\Delta^{-1/2}|\widehat{X}_s - \widehat{X}_{\underline{s}}|^2.
\end{aligned} \tag{2.55}$$

Secondly, by using the property **YW5**, the Condition **B9** and (2.51), we have

$$
\begin{aligned}
J_2(s) &= \frac{1}{2}\phi''_{\delta\varepsilon}(Y_s)\left|\sigma(X_s) - \sigma(\widehat{X}_s) + \sigma(\widehat{X}_s) - \sigma(\widehat{X}_{\underline{s}}) + \sigma(\widehat{X}_{\underline{s}}) - \sigma_\Delta(\widehat{X}_{\underline{s}})\right|^2 \\
&\le \frac{3}{|Y_s|\log\delta}\mathbb{I}_{[\frac{\varepsilon}{\delta};\varepsilon]}(|Y_s|)\left[\left|\sigma(X_s) - \sigma(\widehat{X}_s)\right|^2 + \left|\sigma(\widehat{X}_s) - \sigma(\widehat{X}_{\underline{s}})\right|^2\right. \\
&\quad \left. + \left|\sigma(\widehat{X}_{\underline{s}}) - \sigma_\Delta(\widehat{X}_{\underline{s}})\right|^2\right] \\
&\le \frac{3}{|Y_s|\log\delta}\mathbb{I}_{[\frac{\varepsilon}{\delta};\varepsilon]}(|Y_s|)\left[L_3^2\left(1 + |X_s|^m + |\widehat{X}_s|^m\right)^2|X_s - \widehat{X}_s|^{1+2\alpha}\right. \\
&\quad \left. + L_3^2\left(1 + |\widehat{X}_s|^m + |\widehat{X}_{\underline{s}}|^m\right)^2|\widehat{X}_s - \widehat{X}_{\underline{s}}|^{1+2\alpha} + L_5^2\Delta\left|\sigma(\widehat{X}_{\underline{s}})\right|^4\right] \\
&\le \frac{3}{|Y_s|\log\delta}\mathbb{I}_{[\frac{\varepsilon}{\delta};\varepsilon]}(|Y_s|)\left[3L_3^2\left(1 + |X_s|^{2m} + |\widehat{X}_s|^{2m}\right)|Y_s|^{1+2\alpha} +\right. \\
&\quad \left. + 3L_3^2\left(1 + |\widehat{X}_s|^{2m} + |\widehat{X}_{\underline{s}}|^{2m}\right)|\widehat{X}_s - \widehat{X}_{\underline{s}}|^{1+2\alpha} + L_5^2\Delta\left|\sigma(\widehat{X}_{\underline{s}})\right|^4\right] \\
&\le \frac{9L_3^2\varepsilon^{2\alpha}}{\log\delta}\left(1 + |X_s|^{2m} + |\widehat{X}_s|^{2m}\right) + \frac{3L_5^2\delta\Delta|\sigma(\widehat{X}_{\underline{s}})|^4}{\varepsilon\log\delta} \\
&\quad + \frac{9L_3^2\delta}{\varepsilon\log\delta}\left(1 + C|\widehat{X}_s - \widehat{X}_{\underline{s}}|^{2m} + C|\widehat{X}_{\underline{s}}|^{2m}\right)|\widehat{X}_s - \widehat{X}_{\underline{s}}|^{1+2\alpha} \\
&\le \frac{9L_3^2\varepsilon^{2\alpha}}{\log\delta}\left(1 + |X_s|^{2m} + |\widehat{X}_s|^{2m}\right) + \frac{9L_3^2\delta}{\varepsilon\log\delta}|\widehat{X}_s - \widehat{X}_{\underline{s}}|^{1+2\alpha} \\
&\quad + \frac{9CL_3^2\delta}{\varepsilon\log\delta}|\widehat{X}_s - \widehat{X}_{\underline{s}}|^{1+2\alpha+2m} \\
&\quad + \frac{9CL_3^2\delta}{\varepsilon\log\delta}|\widehat{X}_{\underline{s}}|^{2m}|\widehat{X}_s - \widehat{X}_{\underline{s}}|^{1+2\alpha} + \frac{C\delta\Delta\left(|\widehat{X}_{\underline{s}}|^{2+4\alpha+4m} + 1\right)}{\varepsilon\log\delta}.
\end{aligned} \tag{2.56}
$$

Thirdly, using the mean value theorem, the property **YW2**, the Conditions **B1, B10** and (2.51), there exists an $(\mathcal{F}_s)$-adapted process $\xi = (\xi_s)$ such that

$$
\begin{aligned}
J_3(s) &= \phi_{\delta\varepsilon}\left(Y_s + \left(c\left(X_s\right) - c_\Delta\left(\widehat{X}_{\underline{s}}\right)\right)z\right) - \phi_{\delta\varepsilon}\left(Y_s\right) \\
&\quad - \phi'_{\delta\varepsilon}\left(Y_s\right)\left(c\left(X_s\right) - c_\Delta\left(\widehat{X}_{\underline{s}}\right)\right)z \\
&= \phi'_{\delta\varepsilon}\left(\xi_s\right)\left(c\left(X_s\right) - c_\Delta\left(\widehat{X}_{\underline{s}}\right)\right)z + \phi'_{\delta\varepsilon}\left(Y_s\right)\left(c\left(X_s\right) - c_\Delta\left(\widehat{X}_{\underline{s}}\right)\right)z \\
&\le 2\left|c\left(X_s\right) - c_\Delta\left(\widehat{X}_{\underline{s}}\right)\right||z| \\
&\le 2\left[\left|c\left(X_s\right) - c\left(\widehat{X}_s\right)\right| + \left|c\left(\widehat{X}_s\right) - c\left(\widehat{X}_{\underline{s}}\right)\right| + \left|c\left(\widehat{X}_{\underline{s}}\right) - c_\Delta\left(\widehat{X}_{\underline{s}}\right)\right|\right]|z| \\
&\le 2\left[L_4\left|Y_s\right| + L_4\left|\widehat{X}_s - \widehat{X}_{\underline{s}}\right| + L_5\Delta^{1/2}c^2(\widehat{X}_{\underline{s}})(1 + |b(\widehat{X}_{\underline{s}})|)\right]|z| \\
&\le 2\left[L_4\left|Y_s\right| + L_4\left|\widehat{X}_s - \widehat{X}_{\underline{s}}\right| + C\Delta^{1/2}(1 + |\widehat{X}_{\underline{s}}|^{l+3})\right]|z|.
\end{aligned} \tag{2.57}
$$

By choosing $\lambda = L_1 + 2L_4\mu$ where recall that $\mu = \int_{\mathbb{R}_0} |z|\nu(dz)$ and using **YW3**, we get

$$(L_1 + 2L_4\mu)\left[|Y_s| - \phi_{\delta\varepsilon}(Y_s)\right] \le \varepsilon\left(|L_1| + 2L_4\mu\right). \tag{2.58}$$

A combination of (2.54), (2.55), (2.56), (2.57) and (2.58) implies

$$\begin{aligned}
&\mathbb{E}\left[e^{-(L_1+2L_4\mu)t}|Y_t|\right] \\
&\le e^{-(L_1+2L_4\mu)t}\varepsilon + \int_0^t e^{-(L_1+2L_4\mu)s}\Bigg[\varepsilon\left(|L_1| + 2L_4\right) \\
&\quad + \frac{3}{2}L_2\Delta^{1/2}\left(1 + \mathbb{E}\left[|\widehat{X}_s|^{2l}\right] + \mathbb{E}\left[|\widehat{X}_{\underline{s}}|^{2l}\right]\right) + \frac{1}{2}L_2\Delta^{-1/2}\mathbb{E}\left[|\widehat{X}_s - \widehat{X}_{\underline{s}}|^2\right] \\
&\quad + \frac{9L_3^2\varepsilon^{2\alpha}}{\log\delta}\left(1 + \mathbb{E}\left[|X_s|^{2m}\right] + \mathbb{E}\left[|\widehat{X}_s|^{2m}\right]\right) + \frac{9L_3^2\delta}{\varepsilon\log\delta}\mathbb{E}\left[|\widehat{X}_s - \widehat{X}_{\underline{s}}|^{1+2\alpha}\right] \\
&\quad + \frac{9CL_3^2\delta}{\varepsilon\log\delta}\mathbb{E}\left[|\widehat{X}_s - \widehat{X}_{\underline{s}}|^{1+2\alpha+2m}\right] + \frac{9CL_3^2\delta}{\varepsilon\log\delta}\mathbb{E}\left[|\widehat{X}_{\underline{s}}|^{2m}|\widehat{X}_s - \widehat{X}_{\underline{s}}|^{1+2\alpha}\right] \\
&\quad + \frac{C\delta\Delta\left(\mathbb{E}\left[|\widehat{X}_{\underline{s}}|^{2+4\alpha+4m}\right] + 1\right)}{\varepsilon\log\delta}\Bigg]ds \\
&\quad + \int_0^t\int_{\mathbb{R}_0} 2e^{-(L_1+2L_4\mu)s}\left(L_4\mathbb{E}\left[\left|\widehat{X}_s - \widehat{X}_{\underline{s}}\right|\right] + C\Delta^{1/2}\left(1 + \mathbb{E}\left[|\widehat{X}_{\underline{s}}|^{l+3}\right]\right)\right)|z|\nu(dz)ds \\
&\le e^{-(L_1+2L_4\mu)t}\varepsilon + \int_0^t e^{-(L_1+2L_4\mu)s}\Bigg[\varepsilon\left(|L_1| + 2L_4\right) \\
&\quad + \frac{3}{2}L_2\Delta^{1/2}\left(1 + \mathbb{E}\left[|\widehat{X}_s|^{2l}\right] + \mathbb{E}\left[|\widehat{X}_{\underline{s}}|^{2l}\right]\right) + \frac{1}{2}L_2\Delta^{-1/2}\mathbb{E}\left[|\widehat{X}_s - \widehat{X}_{\underline{s}}|^2\right] \\
&\quad + \frac{9L_3^2\varepsilon^{2\alpha}}{\log\delta}\left(1 + \mathbb{E}\left[|X_s|^{2m}\right] + \mathbb{E}\left[|\widehat{X}_s|^{2m}\right]\right) + \frac{9L_3^2\delta}{\varepsilon\log\delta}\mathbb{E}\left[|\widehat{X}_s - \widehat{X}_{\underline{s}}|^{1+2\alpha}\right] \\
&\quad + \frac{9CL_3^2\delta}{\varepsilon\log\delta}\mathbb{E}\left[|\widehat{X}_s - \widehat{X}_{\underline{s}}|^{1+2\alpha+2m}\right] + \frac{9CL_3^2\delta}{\varepsilon\log\delta}\mathbb{E}\left[|\widehat{X}_{\underline{s}}|^{2m}|\widehat{X}_s - \widehat{X}_{\underline{s}}|^{1+2\alpha}\right] \\
&\quad + \frac{C\delta\Delta\left(\mathbb{E}\left[|\widehat{X}_{\underline{s}}|^{2+4\alpha+4m}\right] + 1\right)}{\varepsilon\log\delta} \\
&\quad + 2\mu\left(L_4\mathbb{E}\left[\left|\widehat{X}_s - \widehat{X}_{\underline{s}}\right|\right] + C\Delta^{1/2}\left(1 + \mathbb{E}\left[|\widehat{X}_{\underline{s}}|^{l+3}\right]\right)\right)\Bigg]ds.
\end{aligned} \tag{2.59}$$

Now, using equation (2.29), Burkholder-Davis-Gundy's inequality, **T3**, **B6**, (2.25), and (2.24), we obtain

$$\begin{aligned}
&\mathbb{E}\left[|\widehat{X}_s - \widehat{X}_{\underline{s}}|^{1+2\alpha}|\mathcal{F}_{\underline{s}}\right] \\
&\le 3^{2\alpha}\Bigg(\mathbb{E}\left[|b(\widehat{X}_{\underline{s}})(s - \underline{s})|^{1+2\alpha}|\mathcal{F}_{\underline{s}}\right] + \mathbb{E}\left[|\sigma_\Delta(\widehat{X}_{\underline{s}})(W_s - W_{\underline{s}})|^{1+2\alpha}|\mathcal{F}_{\underline{s}}\right]
\end{aligned}$$

$$
\begin{aligned}
&\quad + \mathbb{E}\left[|c_\Delta(\widehat{X}_{\underline{s}})(Z_s - Z_{\underline{s}})|^{1+2\alpha}|\mathcal{F}_{\underline{s}}\right]\Bigg) \\
&\leq C\Bigg(|b(\widehat{X}_{\underline{s}})(s-\underline{s})|^{1+2\alpha} + |\sigma_\Delta(\widehat{X}_{\underline{s}})|^{1+2\alpha}(s-\underline{s})^{1/2+\alpha}|c_\Delta(\widehat{X}_{\underline{s}})|^{1+2\alpha}(s-\underline{s})^{1/2+\alpha}\Bigg) \\
&\leq C\Bigg(\left|b(\widehat{X}_{\underline{s}})\right|^{1+2\alpha}\left|h(\widehat{X}_{\underline{s}})\right|^{1+2\alpha}\Delta^{1+2\alpha} + |\sigma_\Delta(\widehat{X}_{\underline{s}})|^{1+2\alpha}\left|h(\widehat{X}_{\underline{s}})\right|^{1/2+\alpha}\Delta^{1/2+\alpha} \\
&\quad + \left|c_\Delta(\widehat{X}_{\underline{s}})\right|^{1+2\alpha}\left|h(\widehat{X}_{\underline{s}})\right|^{1/2+\alpha}\Delta^{1/2+\alpha}\Bigg) \\
&\leq C\Delta^{1/2+\alpha}.
\end{aligned}
$$

This, together with Theorem 2.5.2 and $m \leq p_0/2$, yields that

$$
\begin{aligned}
\mathbb{E}\left[|\widehat{X}_{\underline{s}}|^{2m}|\widehat{X}_s - \widehat{X}_{\underline{s}}|^{1+2\alpha}\right] &= \mathbb{E}\left[\mathbb{E}\left[|\widehat{X}_{\underline{s}}|^{2m}|\widehat{X}_s - \widehat{X}_{\underline{s}}|^{1+2\alpha}|\mathcal{F}_{\underline{s}}\right]\right] \\
&= \mathbb{E}\left[|\widehat{X}_{\underline{s}}|^{2m}\mathbb{E}\left[|\widehat{X}_s - \widehat{X}_{\underline{s}}|^{1+2\alpha}|\mathcal{F}_{\underline{s}}\right]\right] \\
&\leq C\Delta^{1/2+\alpha}\mathbb{E}\left[|\widehat{X}_{\underline{s}}|^{2m}\right] \leq C\Delta^{1/2+\alpha}. \qquad (2.60)
\end{aligned}
$$

Consequently, plugging (2.60) into (2.59), and using condition $p_0 \geq \max\{4l; 2+4\alpha+4m\}$, Theorem 2.5.2, Proposition 2.3.1, and Lemma 2.6.1, for any $T > 0$, there exists a positive constant C_T such that for any $t \in [0, T]$,

$$
\begin{aligned}
&\mathbb{E}\left[e^{-(L_1+2L_4\mu)t}|Y_t|\right] \\
&\leq e^{-(L_1+2L_4\mu)t}\varepsilon + C_T\left[\varepsilon + \Delta^{1/2} + \frac{\varepsilon^{2\alpha}}{\log\delta} + \frac{\delta\Delta^{1/2+\alpha}}{\varepsilon\log\delta} + \frac{\delta\Delta}{\varepsilon\log\delta}\right]\int_0^t e^{-(L_1+2L_4\mu)s}ds. \qquad (2.61)
\end{aligned}
$$

If $\alpha \in \left(0; \frac{1}{2}\right]$, choosing $\varepsilon = \Delta^{1/2}$, $\delta = 2$, we obtain

$$
\sup_{0\leq t\leq T}\mathbb{E}\left[|Y_t|\right] \leq C_T\Delta^\alpha.
$$

If $\alpha = 0$, choosing $\varepsilon = \Delta^{1/4}$, $\delta = \Delta^{-1/4}$, we obtain

$$
\sup_{0\leq t\leq T}\mathbb{E}\left[|Y_t|\right] \leq \frac{C_T}{\log\frac{1}{\Delta}}.
$$

Therefore, we have shown (2.52). Note that if $L_1 + 2L_4\mu < 0$ and $\gamma < 0$, we can choose the constant C_T in (2.61) such that it does not depend on T. Therefore, we also obtain (2.53). This finishes the proof. □

2.7 Numerical experiments

We consider numerical experiments for four different SDEs with coefficients given in Table 2.1. For all equations, $X_0 = 0$, $(Z_t)_{t\geq 0}$ is a compound Poisson process of the form $Z_t = \sum_{i=1}^{N_t} \xi_i$, where $(N_t)_{t\geq 0}$ is a Poisson process with intensity $\lambda = 5$, and $(\xi_i)_{i\geq 1}$ is a sequence of independent and identically distributed random variables. We suppose that each ξ_i has a normal distribution with mean zero and standard deviation 0.2. A simple computation shows that these equations satisfy Conditions **B2**, **B6**–**B10** with constants $p_0, L_1, \gamma, l, m, \alpha$ as given in Table 2.1. In all these cases, $p_0 \geq \max\{4l; 2 + 4\alpha + 4m\}$, hence it follows from Theorem 2.6.2 that the tamed-adaptive Euler-Maruyama approximation scheme defined by (2.23), (2.24) and (2.50) converges in L^1-norm at the rate of order α in any finite time interval. Moreover, in Case 4, since $L_1 < 0$ and $\gamma < 0$, the tamed-adaptive Euler-Maruyama approximation scheme (2.23) converges in L^1-norm at the rate of order α in infinite time intervals.

Case	b	σ	c	p_0	L_1	γ	η	l	m	α
1	$-1+x-x^3$	$1+(1+x)x^{2/3}$	$x+\sin(x)$	10	1	-1	31873	2	$\frac{4}{3}$	$\frac{1}{6}$
2	$-1+x-x^3$	$1+\sqrt{\frac{x^4+x^{4/3}}{14}}$	$x+\sin(x)$	10	1	1	957	2	2	$\frac{1}{6}$
3	$-1-x-x^{7/3}$	$1+\sqrt{\frac{2x^2+x^{10/3}+x^{4/3}}{14}}$	$x+\sin(x)$	10	-1	1	1868	$\frac{4}{3}$	1	$\frac{1}{6}$
4	$-1-x-x^{7/3}$	$1+\sqrt{\frac{x^{10/3}+x^{4/3}}{14}}$	$x+\sin(x)$	10	-1	-1	1583	$\frac{4}{3}$	1	$\frac{1}{6}$

Table 2.1: Four jump SDEs with their parameters

To study the empirical rates of convergence of the tamed-adaptive Euler-Maruyama scheme, we consider

$$me(l) = \frac{1}{M}\sum_{k=1}^{M} |\widehat{X}_5^{(l,k)} - \widehat{X}_5^{(l+1,k)}|,$$

where for each $l \geq 2$, $(\widehat{X}^{(l,k)})_{1\leq k\leq M}$ is a sequence of independent copies of $\widehat{X}^{(l)}$ defined by equations (2.23), (2.24), and (2.50) with $\Delta = 2^{-l}$. We adapt the Algorithm 1 in [18] to generate $\widehat{X}_5^{(l,k)}$ and $\widehat{X}_5^{(l+1,k)}$ on the same paths of the Brownian motion W and the Lévy process Z for each k and l.

If $\widehat{X}^{(l)}$ converges at the rate of order $\beta \in (0, +\infty)$ in L^1-norm, then there exists a constant $\beta > 0$ such that $2^{\beta l}\mathbb{E}[|X_5 - \widehat{X}_5^{(l)}|] = O(1)$, implying that $2^{\beta l}\mathbb{E}[|\widehat{X}_5^{(l+1)} - \widehat{X}_5^{(l)}|] = O(1)$ and vice-versa. In this case, we can write $\log_2 me(l) = -\beta l + C + o(1)$, for some constant $C \in \mathbb{R}$. Thus β can be estimated by the regression method.

Figure 2.1 shows the simulation result of $\log_2 me(l)$ for $l = 2, ..., 6$. We draw the regression lines to estimate the empirical rates of convergence β in each case. In Case 2, the empirical rate of convergence, which is 0.1819, is almost the same as the theoretical rate, which is $1/6$. In the other cases, the empirical rates are slightly better than the theoretical rate.

Note that in Case 4, the tamed-adaptive Euler-Maruyama approximation converges in infinite time intervals while in other cases, it converges in any finite time intervals.

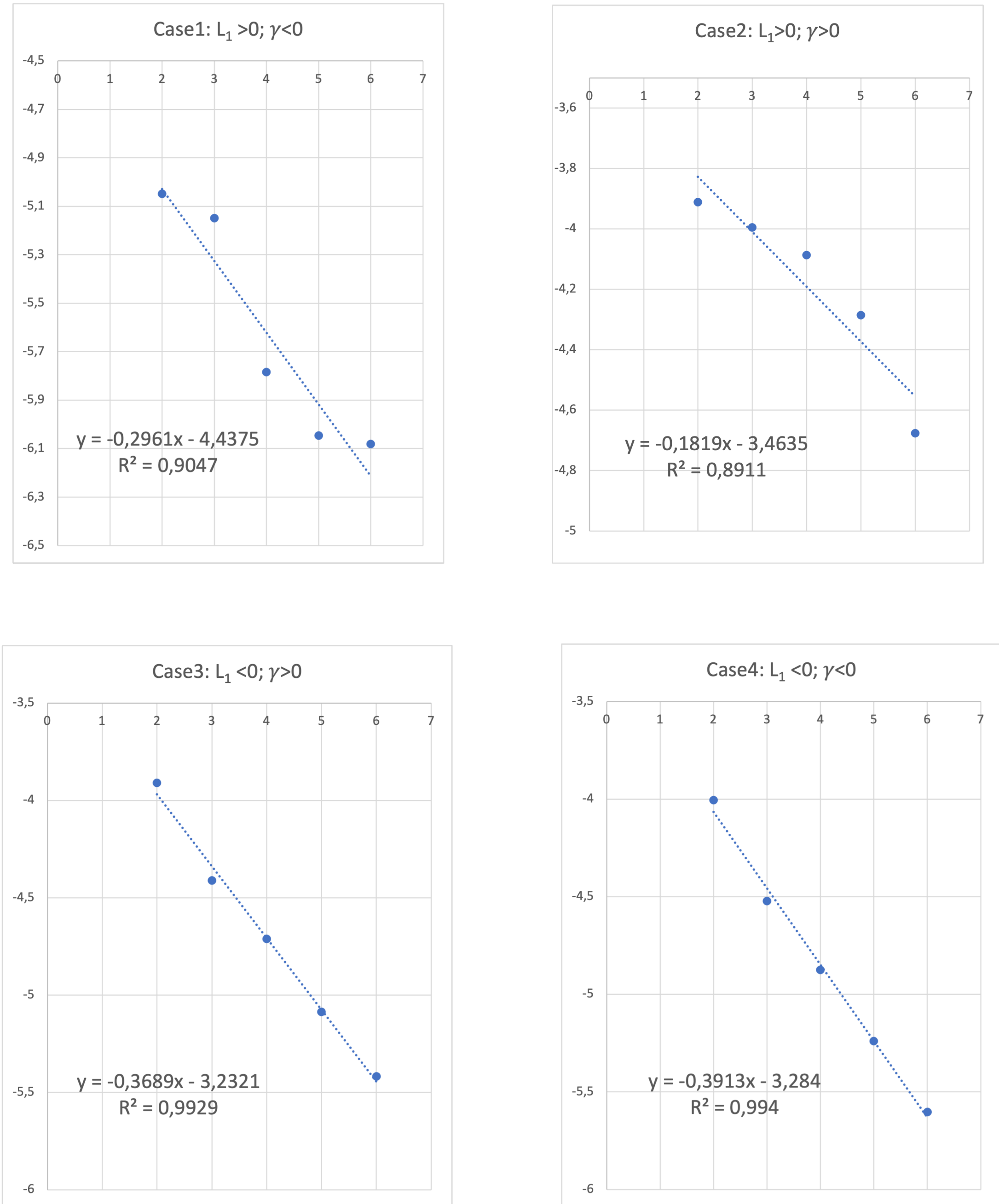

Figure 2.1: Values of $\log_2(me(l))$ for $l = 2, 3, 4, 5, 6$.

2.8 Conclusions

In this chapter, we proposed and analyzed a novel tamed-adaptive Euler-Maruyama (TAEM) scheme for Lévy-driven SDEs (2.1). This scheme is specifically designed to handle the simultaneous presence of multiple coefficient irregularities: (i) a locally Hölder continuous diffusion coefficient (σ), and (ii) super-linear growth in both the drift (b) and diffusion (σ) coefficients.

We established the strong convergence of this scheme, proving that it achieves the expected convergence rate (order α). Crucially, our analysis confirms this convergence not only on finite time intervals $[0;T]$, but also on the infinite time horizon $[0;\infty)$ under standard stability conditions ($\gamma < 0$, $L_1 + 2L_4\mu < 0$).

This finding represents a significant generalization of existing works. It extends the adaptive framework of Fang and Giles [18], which was designed for Brownian-driven SDEs with Lipschitz diffusion, to the non-Lipschitz (Hölder) setting. Furthermore, it directly generalizes our own prior work [44] (which handled the Hölder continuous, Brownian-driven case) by successfully incorporating the non-trivial jump component. This extension required a careful reformulation of the key moment-controlling assumption (Condition **B2**) and a rigorous analysis of the interactions between the jumps, the Hölder diffusion, and the adaptive step-sizing mechanism.

Chapter 3

TAMED-ADAPTIVE EULER-MARUYAMA SCHEME FOR LÉVY-DRIVEN MCKEAN-VLASOV SDEs WITH IRREGULAR COEFFICIENTS

Following the analysis of the tamed-adaptive Euler-Maruyama scheme for stochastic differential equations with jumps in Chapter 2, this chapter extends the investigation to the numerical approximation of a more complex class of equations, namely McKean-Vlasov SDEs with jumps. The fundamental challenge of these equations is that their coefficients depend on both the state and the probability distribution of the process, creating a complex mean-field interaction structure. In this chapter, we focus on the case where the drift and diffusion coefficients are non-globally Lipschitz continuous and exhibit superlinear growth. The results presented herein are based on the author's publication [3], listed in the **List of Author's Related Papers** section.

3.1 Introduction

On a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, we consider the d-dimensional process $X = (X_t)_{t\geq 0}$ is the solution to the following McKean-Vlasov stochastic differential equation (SDE) with jumps

$$dX_t = b(X_t, \mathcal{L}_{X_t})dt + \sigma(X_t, \mathcal{L}_{X_t})dW_t + c\left(X_{t-}, \mathcal{L}_{X_{t-}}\right) dZ_t, \tag{3.1}$$

for $t \geq 0$, where $X_0 = x_0 \in \mathbb{R}^d$ is a fixed initial value, $\mathcal{L}_{X_t}$ denotes the marginal law of the process X at time t, $W = (W_t)_{t\geq 0}$ is a d-dimensional standard Brownian motion and $Z = (Z_t)_{t\geq 0}$ is a d-dimensional centered pure jump Lévy process whose Lévy measure ν satisfies $\int_{\mathbb{R}^d}(1 \wedge |z|^2)\nu(dz) < +\infty$. The processes W and Z are assumed to be independent. The natural filtration $(\mathcal{F}_t)_{t\geq 0}$ is generated by W and Z.

We denote by $\mathcal{P}(\mathbb{R}^d)$ the space of all probability measures defined on a measurable space $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$, where $\mathcal{B}(\mathbb{R}^d)$ denotes the Borel σ-field over $\mathbb{R}^d$, and by

$$\mathcal{P}_2(\mathbb{R}^d) := \left\{ \mu \in \mathcal{P}(\mathbb{R}^d) : \int_{\mathbb{R}^d} |x|^2 \mu(dx) < \infty \right\}$$

the subset of probability measures with finite second moment. The space $\mathcal{P}_2(\mathbb{R}^d)$ is equipped with the $\mathcal{L}_2$-Wasserstein distance. That is, for $\mu, \nu \in \mathcal{P}_2(\mathbb{R}^d)$, the $\mathcal{L}_2$-Wasserstein distance between μ and ν is defined as

$$\mathcal{W}_2(\mu, \nu) := \inf_{\pi \in \mathcal{C}(\mu,\nu)} \left(\int_{\mathbb{R}^d \times \mathbb{R}^d} |x - y|^2 \, \pi(dx, dy) \right)^{1/2},$$

where $\mathcal{C}(\mu, \nu)$ denotes all the couplings of μ and ν, i.e., $\pi \in \mathcal{C}(\mu, \nu)$ if and only if $\pi(\cdot, \mathbb{R}^d) = \mu(\cdot)$ and $\pi(\mathbb{R}^d, \cdot) = \nu(\cdot)$.

The coefficients $b = (b_i)_{1 \le i \le d} : \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}^d$, $\sigma = (\sigma_{ij})_{1 \le i,j \le d} : \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}^d \otimes \mathbb{R}^d$ and $c = (c_{ij})_{1 \le i,j \le d} : \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}^d \otimes \mathbb{R}^d$ are measurable functions. The integral equation (3.1) can be written as

$$X_t = x_0 + \int_0^t b(X_s, \mathcal{L}_{X_s}) ds + \int_0^t \sigma(X_s, \mathcal{L}_{X_s}) dW_s + \int_0^t \int_{\mathbb{R}_0^d} c\left(X_{s-}, \mathcal{L}_{X_{s-}}\right) z \widetilde{N}(ds, dz),$$

for any $t \ge 0$.

The McKean-Vlasov process was first studied by McKean in [65] as a model for the Vlasov equation describing the time evolution of the distribution function of a plasma consisting of charged particles with long-range interaction. The process can be obtained as a limit of a mean-field system of interacting particles as the number of particles tends to infinity. The very first studies on the numerical approximation for McKean-Vlasov SDEs are the works of Ogawa [74], Kohatsu-Higa and Ogawa [47] and Bossy and Talay [5], where the authors considered the weak approximation of McKean-Vlasov SDEs with regular coefficients. However, the numerical approximation for McKean-Vlasov SDEs has only become active in the last decade.

Let $X_T^{(n)}$ be an approximation of X_T which depends on the values of W and Z at fixed equidistant times $t_k = \frac{kT}{n}, k = 0, 1, \ldots, n$. Then under some regularity conditions on the coefficients b, σ, and c, one may prove that

$$|X_T^{(n)} - X_T|_{L^p} := \left(\mathbb{E}[|X_T^{(n)} - X_T|^p] \right)^{1/p} \le \frac{C(T)}{n^{\zeta_0}}, \tag{3.2}$$

for some positive constants p and ζ_0. If the estimate (3.2) holds, we say that the $X_T^{(n)}$ converges at the rate of order ζ_0 in L^p-norm.

In [15, 50, 51, 60] the tamed Euler-Maruyama scheme has been developed to approximate McKean-Vlasov SDEs with super-linear growth coefficients. In [77], the authors introduced several adaptive Euler-Maruyama and Milstein schemes and studied their strong rate of convergence for McKean-Vlasov SDEs with super-linear drift. In [38], the authors introduced a multilevel Picard approximation, which has a low computational cost, for McKean-Vlasov SDEs with additive noise. In [8, 15], the authors introduced the implicit Euler-Maruyama scheme and studied its convergence in L^p-norm for McKean-Vlasov SDEs with drifts of super-linear growth.

The McKean-Vlasov SDEs with jumps have been studied by many authors with applications in many domains [1, 16, 17, 20, 24] and the references therein. In [68], the authors considered McKean-Vlasov SDEs driven by infinite activity Lévy processes with super-linear growth coefficients. They proved the existence and uniqueness of the solution and proposed a tamed Euler-Maruyama approximation for the associated interacting particle system and proved that the rate of its convergence in L^p-norm is arbitrarily close to 0.5.

3.2 Model assumptions

In this chapter, we assume that the drift, diffusion and jump coefficients b, σ, c and the Lévy measure ν of equation (3.1) satisfy the following conditions:

C1. There exists a positive constant L such that

$$2\langle x, b(x,\mu)\rangle + |\sigma(x,\mu)|^2 + |c(x,\mu)|^2 \int_{\mathbb{R}_0^d} |z|^2 \nu(dz) \leq L\left(1 + |x|^2 + \mathcal{W}_2^2(\mu, \delta_0)\right),$$

for any $x \in \mathbb{R}^d$ and $\mu \in \mathcal{P}_2(\mathbb{R}^d)$.

C2. There exist constants $\kappa_1 > 0, \kappa_2 > 0$, $L_1 \in \mathbb{R}$ and $L_2 \geq 0$ such that

$$\begin{aligned} &2\langle x - \overline{x}, b(x,\mu) - b(\overline{x},\overline{\mu})\rangle + \kappa_1 |\sigma(x,\mu) - \sigma(\overline{x},\overline{\mu})|^2 \\ &+ \kappa_2 |c(x,\mu) - c(\overline{x},\overline{\mu})|^2 \int_{\mathbb{R}_0^d} |z|^2 \nu(dz) \leq L_1 |x - \overline{x}|^2 + L_2 \mathcal{W}_2^2(\mu, \overline{\mu}), \end{aligned}$$

for any $x, \overline{x} \in \mathbb{R}^d$ and $\mu, \overline{\mu} \in \mathcal{P}_2(\mathbb{R}^d)$.

C3. $b(x,\mu)$ is a continuous function of $x \in \mathbb{R}^d$ and $\mu \in \mathcal{P}_2(\mathbb{R}^d)$.

C4. There exist constants $L > 0$ and $\ell \geq 1$ such that

$$|b(x,\mu) - b(\overline{x},\overline{\mu})| \leq L\left(1 + |x|^\ell + |\overline{x}|^\ell\right)\left(|x - \overline{x}| + \mathcal{W}_2(\mu,\overline{\mu})\right),$$

for any $x, \overline{x} \in \mathbb{R}^d$ and $\mu, \overline{\mu} \in \mathcal{P}_2(\mathbb{R}^d)$.

C5. There exists an even integer $p_0 \in [2; +\infty)$ such that $\int_{|z|>1} |z|^{p_0} \nu(dz) < \infty$ and $\int_{0<|z|\leq 1} |z|\nu(dz) < \infty$.

C6. There exists a positive constant L_3 such that

$$|c(x, \mu)| \leq L_3 \left(1 + |x| + \mathcal{W}_2(\mu, \delta_0)\right),$$

for any $x \in \mathbb{R}^d$ and $\mu \in \mathcal{P}_2(\mathbb{R}^d)$.

C7. For the even integer $p_0 \in [2; +\infty)$ given in **C5** and the positive constant L_3 given in **C6**, there exist constants $\gamma_1 \in \mathbb{R}$, $\gamma_2 \geq 0$ and $\eta \geq 0$ such that

$$\langle x, b(x, \mu)\rangle + \frac{p_0 - 1}{2}|\sigma(x, \mu)|^2 + \frac{1}{2L_3}|c(x, \mu)|^2 \int_{\mathbb{R}_0^d} |z| \left((1 + L_3|z|)^{p_0-1} - 1\right) \nu(dz)$$
$$\leq \gamma_1|x|^2 + \gamma_2\mathcal{W}_2^2(\mu, \delta_0) + \eta,$$

for any $x \in \mathbb{R}^d$ and $\mu \in \mathcal{P}_2(\mathbb{R}^d)$.

Remark 3.2.1. (i) It follows from Condition **C4** that

$$|b(x, \mu)| \leq |b(0, \delta_0)| + L\left(1 + |x|^\ell\right)\left(|x| + \mathcal{W}_2(\mu, \delta_0)\right),$$

for any $x \in \mathbb{R}^d$ and $\mu \in \mathcal{P}_2(\mathbb{R}^d)$.

(ii) Assume that Condition **C2** holds for $\kappa_1 = \kappa_2 = 1 + \varepsilon$, $L_1 \in \mathbb{R}$, $L_2 \geq 0$ with a constant $\varepsilon > 0$. This, combined with Condition **C4** and Cauchy's inequality, implies that

$$\begin{aligned}
&(1 + \varepsilon)\left|\sigma(x, \mu) - \sigma(\overline{x}, \overline{\mu})\right|^2 + (1 + \varepsilon)\left|c(x, \mu) - c(\overline{x}, \overline{\mu})\right|^2 \int_{\mathbb{R}_0^d} |z|^2\nu(dz) \\
&\leq 2|x - \overline{x}||b(x, \mu) - b(\overline{x}, \overline{\mu})| + L_1|x - \overline{x}|^2 + L_2\mathcal{W}_2^2(\mu, \overline{\mu}) \\
&\leq L\left(1 + |x|^\ell + |\overline{x}|^\ell\right)\left(2|x - \overline{x}|^2 + 2|x - \overline{x}|\mathcal{W}_2(\mu, \overline{\mu})\right) + L_1|x - \overline{x}|^2 + L_2\mathcal{W}_2^2(\mu, \overline{\mu}) \\
&\leq L\left(1 + |x|^\ell + |\overline{x}|^\ell\right)\left(2|x - \overline{x}|^2 + |x - \overline{x}|^2 + \mathcal{W}_2^2(\mu, \overline{\mu})\right) + L_1|x - \overline{x}|^2 + L_2\mathcal{W}_2^2(\mu, \overline{\mu}) \\
&\leq L\widetilde{L}\left(1 + |x|^\ell + |\overline{x}|^\ell\right)\left(|x - \overline{x}|^2 + \mathcal{W}_2^2(\mu, \overline{\mu})\right),
\end{aligned}$$

with $\widetilde{L} = \max\{3, L_1, L_2\}$ for any $x, \overline{x} \in \mathbb{R}^d$ and $\mu, \overline{\mu} \in \mathcal{P}_2(\mathbb{R}^d)$. This yields

$$\begin{aligned}
&\left|\sigma(x, \mu) - \sigma(\overline{x}, \overline{\mu})\right|^2 + \left|c(x, \mu) - c(\overline{x}, \overline{\mu})\right|^2 \int_{\mathbb{R}_0^d} |z|^2\nu(dz) \\
&\leq \frac{L\widetilde{L}}{1 + \varepsilon}\left(1 + |x|^\ell + |\overline{x}|^\ell\right)\left(|x - \overline{x}|^2 + \mathcal{W}_2^2(\mu, \overline{\mu})\right),
\end{aligned}$$

for any $x, \overline{x} \in \mathbb{R}^d$ and $\mu, \overline{\mu} \in \mathcal{P}_2(\mathbb{R}^d)$.

(iii) From (ii), we have that for any $x \in \mathbb{R}^d$ and $\mu \in \mathcal{P}_2(\mathbb{R}^d)$,

$$\begin{aligned}|\sigma(x,\mu)|^2 &\leq 2\,|\sigma(x,\mu)-\sigma(0,\delta_0)|^2 + 2\,|\sigma(0,\delta_0)|^2 \\ &\leq 2\frac{L\widetilde{L}}{1+\varepsilon}\left(1+|x|^\ell\right)\left(|x|^2+\mathcal{W}_2^2(\mu,\delta_0)\right) + 2\,|\sigma(0,\delta_0)|^2,\end{aligned}$$

and similarly,

$$\begin{aligned}|c(x,\mu)|^2 \int_{\mathbb{R}_0^d} |z|^2\nu(dz) &\leq 2\frac{L\widetilde{L}}{1+\varepsilon}\left(1+|x|^\ell\right)\left(|x|^2+\mathcal{W}_2^2(\mu,\delta_0)\right) \\ &\quad + 2\,|c(0,\delta_0)|^2 \int_{\mathbb{R}_0^d} |z|^2\nu(dz).\end{aligned}$$

Remark 3.2.2. Observe that Condition **C7** yields to

$$\langle x, b(x,\mu)\rangle + \frac{p-1}{2}|\sigma(x,\mu)|^2 + \frac{1}{2L_3}|c(x,\mu)|^2 \int_{\mathbb{R}_0^d} |z|\left((1+L_3|z|)^{p-1}-1\right)\nu(dz)$$
$$\leq \gamma_1|x|^2 + \gamma_2\mathcal{W}_2^2(\mu,\delta_0) + \eta,$$

for any $p \in [2; p_0]$, $x \in \mathbb{R}^d$ and $\mu \in \mathcal{P}_2(\mathbb{R}^d)$.

3.3 Lévy-driven McKean-Vlasov SDEs with irregular coefficients

In this section, we present results concerning the exact solution of the McKean-Vlasov stochastic differential equation with jumps (3.1). We begin by recalling a key result on the existence and uniqueness of the strong solution for this SDE, established in [68].

Proposition 3.3.1 ([68], Theorem 2.1)**.** *Assume Conditions **C1**, **C3** hold and Condition **C2** holds for $\kappa_1 = \kappa_2 = 1, L_1 = L_2 > 0$. Then, there exists a unique càdlàg process $X = (X_t)_{t\geq 0}$ taking values in $\mathbb{R}^d$ satisfying the McKean-Vlasov SDE with jumps (3.1) such that*

$$\sup_{t\in[0,T]} \mathbb{E}\left[|X_t|^2\right] \leq K,$$

where $T > 0$ is a fixed constant and $K := K(|x_0|^2, d, L, L_1, T)$ is a positive constant.

Next, we demonstrate the following moment estimates for the exact solution $X = (X_t)_{t\geq 0}$ of the McKean-Vlasov stochastic differential equation with jumps (3.1).

Proposition 3.3.2. *Let $X = (X_t)_{t\geq 0}$ be a solution to equation (3.1). Assume Conditions **C6**, **C7** hold, σ is bounded on $C \times \mathcal{P}_2(\mathbb{R}^d)$ for every compact subset C of $\mathbb{R}^d$,* *and **C5** holds for $q = 2p_0$. Then for any $p \in [2; p_0]$, there exists a positive constant C_p such that for any $t \geq 0$*

$$\mathbb{E}\left[|X_t|^p\right] \leq \begin{cases} C_p(1 + e^{\gamma p t}) & \text{if } \ \gamma \neq 0, \\ C_p(1+t)^{p/2} & \text{if } \ \gamma = 0, p = 2 \ \text{ or } \ \gamma = 0, \gamma_2 > 0, p \in (2, p_0], \\ C_p(1+t)^p & \text{if } \ \gamma = 0, \gamma_2 = 0, p \in (2, p_0], \end{cases} \tag{3.3}$$

where $\gamma = \gamma_1 + \gamma_2$.

Note that if $\gamma < 0$, we have that $\sup_{t \geq 0} \mathbb{E}\left[|X_t|^p\right] \leq 2C_p$.

Proof. Step 1: We first show that for any even natural number $p \in [2; p_0]$ and $T > 0$,

$$\sup_{t \in [0,T]} \mathbb{E}\left[|X_t|^p\right] \leq C(T, p). \tag{3.4}$$

Note that (3.4) holds for $p = 2$ due to Proposition 3.3.1. Next, we assume that (3.4) is valid for any even natural number $q \in [2; p-2]$. That is,

$$\sup_{t \in [0,T]} \mathbb{E}\left[|X_t|^q\right] \leq C(T, q). \tag{3.5}$$

Now, for $\lambda \in \mathbb{R}$ and even natural number $p \in [2, p_0]$, applying Itô's formula to $e^{-\lambda t}|X_t|^p$, we obtain that for any $t \geq 0$,

$$\begin{aligned} & e^{-\lambda t}|X_t|^p \\ &= |x_0|^p + \int_0^t e^{-\lambda s}\Big[-\lambda |X_s|^p + p|X_s|^{p-2}\langle X_s, b(X_s, \mathcal{L}_{X_s})\rangle + \frac{p}{2}|X_s|^{p-2}|\sigma(X_s, \mathcal{L}_{X_s})|^2 \\ &\quad + \frac{p(p-2)}{2}|X_s|^{p-4}\left|X_s^{\mathsf{T}}\sigma(X_s, \mathcal{L}_{X_s})\right|^2\Big] ds + p\int_0^t e^{-\lambda s}|X_s|^{p-2}\langle X_s, \sigma(X_s, \mathcal{L}_{X_s})\, dW_s\rangle \\ &\quad + \int_0^t \int_{\mathbb{R}_0^d} e^{-\lambda s}\left(\left|X_{s-} + c\left(X_{s-}, \mathcal{L}_{X_{s-}}\right) z\right|^p - |X_{s-}|^p\right) \widetilde{N}(ds, dz) \\ &\quad + \int_0^t \int_{\mathbb{R}_0^d} e^{-\lambda s}\left(|X_s + c(X_s, \mathcal{L}_{X_s}) z|^p - |X_s|^p - p|X_s|^{p-2}\langle X_s, c(X_s, \mathcal{L}_{X_s}) z\rangle\right) \nu(dz) ds. \end{aligned} \tag{3.6}$$

In order to treat the last integral in (3.6), it suffices to use the binomial theorem to get that, for any $t \geq 0$,

$$\begin{aligned} |X_s + c(X_s, \mathcal{L}_{X_s}) z|^p &= \left(|X_s + c(X_s, \mathcal{L}_{X_s}) z|^2\right)^{p/2} \\ &= \left(|X_s|^2 + |c(X_s, \mathcal{L}_{X_s}) z|^2 + 2\langle X_s, c(X_s, \mathcal{L}_{X_s}) z\rangle\right)^{p/2} \end{aligned}$$

$$
\begin{aligned}
&= |X_s|^p + \frac{p}{2}|X_s|^{p-2}\left(|c\left(X_s,\mathcal{L}_{X_s}\right)z|^2 + 2\langle X_s, c\left(X_s,\mathcal{L}_{X_s}\right)z\rangle\right)\\
&\quad + \sum_{i=2}^{p/2}\binom{p/2}{i}|X_s|^{p-2i}\left(|c\left(X_s,\mathcal{L}_{X_s}\right)z|^2 + 2\left\langle X_s, c\left(X_s,\mathcal{L}_{X_s}\right)z\right\rangle\right)^i\\
&= |X_s|^p + p|X_s|^{p-2}\left\langle X_s, c\left(X_s,\mathcal{L}_{X_s}\right)z\right\rangle + \frac{p}{2}|X_s|^{p-2}\left|c\left(X_s,\mathcal{L}_{X_s}\right)z\right|^2\\
&\quad + \sum_{i=2}^{p/2}\binom{p/2}{i}|X_s|^{p-2i}\left(|c\left(X_s,\mathcal{L}_{X_s}\right)z|^2 + 2\left\langle X_s, c\left(X_s,\mathcal{L}_{X_s}\right)z\right\rangle\right)^i. \qquad (3.7)
\end{aligned}
$$

Next, by using the binomial theorem repeatedly, Condition **C6**, the equality $\mathcal{W}_2^2\left(\mathcal{L}_{X_s},\delta_0\right) = \mathbb{E}[|X_s|^2]$, and the inequality $y|x|^{p-3} \leq \frac{1}{2}(|x|^{p-2} + y^2|x|^{p-4})$ valid for any $x \in \mathbb{R}^d, y > 0$, $\sum_{j=0}^{i}\binom{i}{j}a^j = (1+a)^i$ and $\sum_{j=0}^{i}\binom{i}{j}ja^j = ia(1+a)^{i-1}$ valid for any $a \in \mathbb{R}$, we obtain that

$$
\begin{aligned}
&|X_s|^{p-2i}\left(|c\left(X_s,\mathcal{L}_{X_s}\right)z|^2 + 2\left\langle X_s, c\left(X_s,\mathcal{L}_{X_s}\right)z\right\rangle\right)^i\\
&= |X_s|^{p-2i}\sum_{j=0}^{i}\binom{i}{j}|c\left(X_s,\mathcal{L}_{X_s}\right)z|^{2i-2j}2^j\left(\left\langle X_s, c\left(X_s,\mathcal{L}_{X_s}\right)z\right\rangle\right)^j\\
&\leq \sum_{j=0}^{i}\binom{i}{j}2^j|X_s|^{p-2i+j}|c\left(X_s,\mathcal{L}_{X_s}\right)|^{2i-j}|z|^{2i-j}\\
&= |c\left(X_s,\mathcal{L}_{X_s}\right)|^2\sum_{j=0}^{i}\binom{i}{j}2^j|X_s|^{p-2i+j}|c\left(X_s,\mathcal{L}_{X_s}\right)|^{2i-j-2}|z|^{2i-j}\\
&\leq |c\left(X_s,\mathcal{L}_{X_s}\right)|^2\sum_{j=0}^{i}\binom{i}{j}2^j|X_s|^{p-2i+j}|z|^{2i-j}L_3^{2i-j-2}\left(1+|X_s|+\mathcal{W}_2(\mathcal{L}_{X_s},\delta_0)\right)^{2i-j-2}\\
&= |c\left(X_s,\mathcal{L}_{X_s}\right)|^2\sum_{j=0}^{i}\binom{i}{j}2^j|X_s|^{p-2i+j}|z|^{2i-j}L_3^{2i-j-2}\left(1+|X_s|+\sqrt{\mathbb{E}[|X_s|^2]}\right)^{2i-j-2}\\
&= |c\left(X_s,\mathcal{L}_{X_s}\right)|^2\sum_{j=0}^{i}\binom{i}{j}2^j|X_s|^{p-2i+j}|z|^{2i-j}L_3^{2i-j-2}\Bigg[|X_s|^{2i-j-2} + (2i-j-2)|X_s|^{2i-j-3}\\
&\quad \times\left(1+\sqrt{\mathbb{E}[|X_s|^2]}\right) + \sum_{k=2}^{2i-j-2}\binom{2i-j-2}{k}\left(1+\sqrt{\mathbb{E}[|X_s|^2]}\right)^k|X_s|^{2i-j-2-k}\Bigg]\\
&= |c\left(X_s,\mathcal{L}_{X_s}\right)|^2\sum_{j=0}^{i}\binom{i}{j}2^j|z|^{2i-j}L_3^{2i-j-2}\Bigg[|X_s|^{p-2} + (2i-j-2)|X_s|^{p-3}\\
&\quad \times\left(1+\sqrt{\mathbb{E}[|X_s|^2]}\right) + \sum_{k=2}^{2i-j-2}\binom{2i-j-2}{k}\left(1+\sqrt{\mathbb{E}[|X_s|^2]}\right)^k|X_s|^{p-2-k}\Bigg]
\end{aligned}
$$

$$
\begin{aligned}
&\leq |c(X_s,\mathcal{L}_{X_s})|^2\sum_{j=0}^{i}\binom{i}{j}2^j|z|^{2i-j}L_3^{2i-j-2}\Bigg[|X_s|^{p-2}+\frac{(2i-j-2)}{2}\Bigg(|X_s|^{p-2}\\
&\quad+\left(1+\sqrt{\mathbb{E}[|X_s|^2]}\right)^2|X_s|^{p-4}\Bigg)+\sum_{k=2}^{2i-j-2}\binom{2i-j-2}{k}\left(1+\sqrt{\mathbb{E}[|X_s|^2]}\right)^k|X_s|^{p-2-k}\Bigg]\\
&=|c(X_s,\mathcal{L}_{X_s})|^2|X_s|^{p-2}\sum_{j=0}^{i}\binom{i}{j}2^j|z|^{2i-j}L_3^{2i-j-2}\left(i-\frac{j}{2}\right)\\
&\quad+|c(X_s,\mathcal{L}_{X_s})|^2\sum_{j=0}^{i}\binom{i}{j}2^j|z|^{2i-j}L_3^{2i-j-2}\Bigg[\left(i-\frac{j}{2}-1\right)\left(1+\sqrt{\mathbb{E}[|X_s|^2]}\right)^2|X_s|^{p-4}\\
&\quad+\sum_{k=2}^{2i-j-2}\binom{2i-j-2}{k}\left(1+\sqrt{\mathbb{E}[|X_s|^2]}\right)^k|X_s|^{p-2-k}\Bigg]\\
&=|c(X_s,\mathcal{L}_{X_s})|^2|X_s|^{p-2}\left[\frac{i}{L_0^2}\left(L_0^2|z|^2+2L_0|z|\right)^i-\frac{|z|}{L_0}i\left(L_0^2|z|^2+2L_0|z|\right)^{i-1}\right]\\
&\quad+|c(X_s,\mathcal{L}_{X_s})|^2\sum_{j=0}^{i}\binom{i}{j}2^j|z|^{2i-j}L_3^{2i-j-2}\Bigg[\left(i-\frac{j}{2}-1\right)\left(1+\sqrt{\mathbb{E}[|X_s|^2]}\right)^2|X_s|^{p-4}\\
&\quad+\sum_{k=2}^{2i-j-2}\binom{2i-j-2}{k}\left(1+\sqrt{\mathbb{E}[|X_s|^2]}\right)^k|X_s|^{p-2-k}\Bigg]\\
&\leq|c(X_s,\mathcal{L}_{X_s})|^2|X_s|^{p-2}\left[\frac{i}{L_0^2}\left(L_0^2|z|^2+2L_0|z|\right)^i-\frac{|z|}{L_0}i\left(L_0^2|z|^2+2L_0|z|\right)^{i-1}\right]\\
&\quad+L_3^2\left(1+|X_s|+\sqrt{\mathbb{E}[|X_s|^2]}\right)^2\sum_{j=0}^{i}\binom{i}{j}2^j|z|^{2i-j}L_3^{2i-j-2}\Bigg[\left(i-\frac{j}{2}-1\right)\\
&\quad\times\left(1+\sqrt{\mathbb{E}[|X_s|^2]}\right)^2|X_s|^{p-4}+\sum_{k=2}^{2i-j-2}\binom{2i-j-2}{k}\left(1+\sqrt{\mathbb{E}[|X_s|^2]}\right)^k|X_s|^{p-2-k}\Bigg]\\
&=|c(X_s,\mathcal{L}_{X_s})|^2|X_s|^{p-2}\frac{i|z|}{L_0}\left(1+L_0|z|\right)\left(L_0^2|z|^2+2L_0|z|\right)^{i-1}\\
&\quad+\sum_{j=0}^{i}|z|^{2i-j}Q_p\left(p-2,2i-j,|X_s|,1+\sqrt{\mathbb{E}[|X_s|^2]}\right),
\end{aligned}
$$

where $2i-j\geq 2$ and

$$
Q_q(m,n,x,y)=\sum_{\ell_1\leq m,\ell_2\leq n,\ell_1+\ell_2=q}c_{\ell_1\ell_2}x^{\ell_1}y^{\ell_2}.
$$

This, together with the fact that $\sum_{i=2}^{p/2}\binom{p/2}{i}ia^{i-1}=\frac{p}{2}((1+a)^{p/2-1}-1)$ valid for any

$a \in \mathbb{R}$, yields to

$$\sum_{i=2}^{p/2} \binom{p/2}{i} |X_s|^{p-2i} \left(|c(X_s, \mathcal{L}_{X_s}) z|^2 + 2 \langle X_s, c(X_s, \mathcal{L}_{X_s}) z \rangle \right)^i$$
$$\leq \frac{|z|}{L_3} (1 + L_0|z|) |c(X_s, \mathcal{L}_{X_s})|^2 |X_s|^{p-2} \sum_{i=2}^{p/2} \binom{p/2}{i} i \left(L_0^2 |z|^2 + 2L_0|z| \right)^{i-1}$$
$$+ \sum_{i=2}^{p/2} \sum_{j=0}^{i} \binom{p/2}{i} |z|^{2i-j} Q_p \left(p-2, 2i-j, |X_s|, 1 + \sqrt{\mathbb{E}[|X_s|^2]} \right)$$
$$= \frac{p}{2} |X_s|^{p-2} |c(X_s, \mathcal{L}_{X_s})|^2 \frac{|z|}{L_3} \left((1 + L_0|z|)^{p-1} - L_3|z| - 1 \right)$$
$$+ \sum_{i=2}^{p/2} \sum_{j=0}^{i} \binom{p/2}{i} |z|^{2i-j} Q_p \left(p-2, 2i-j, |X_s|, 1 + \sqrt{\mathbb{E}[|X_s|^2]} \right). \tag{3.8}$$

As a consequence of (3.7) and (3.8), we have shown that for any $s \geq 0$,

$$|X_s + c(X_s, \mathcal{L}_{X_s}) z|^p - |X_s|^p - p|X_s|^{p-2} \langle X_s, c(X_s, \mathcal{L}_{X_s}) z \rangle$$
$$\leq \frac{p}{2} |X_s|^{p-2} |c(X_s, \mathcal{L}_{X_s})|^2 \left(|z|^2 + \frac{|z|}{L_3} \left((1 + L_0|z|)^{p-1} - L_3|z| - 1 \right) \right)$$
$$+ \sum_{i=2}^{p/2} \sum_{j=0}^{i} \binom{p/2}{i} |z|^{2i-j} Q_p \left(p-2, 2i-j, |X_s|, 1 + \sqrt{\mathbb{E}[|X_s|^2]} \right)$$
$$= \frac{p}{2L_3} |X_s|^{p-2} |c(X_s, \mathcal{L}_{X_s})|^2 |z| \left((1 + L_0|z|)^{p-1} - 1 \right)$$
$$+ \sum_{i=2}^{p/2} \sum_{j=0}^{i} \binom{p/2}{i} |z|^{2i-j} Q_p \left(p-2, 2i-j, |X_s|, 1 + \sqrt{\mathbb{E}[|X_s|^2]} \right). \tag{3.9}$$

Therefore, substituting (3.9) into (3.6), we get that for any $t \geq 0$,

$$e^{-\lambda t} |X_t|^p$$
$$\leq |x_0|^p + p \int_0^t e^{-\lambda s} |X_s|^{p-2} \Bigg[-\frac{\lambda}{p} |X_s|^2 + \langle X_s, b(X_s, \mathcal{L}_{X_s}) \rangle$$
$$+ \frac{p-1}{2} |\sigma(X_s, \mathcal{L}_{X_s})|^2 + \frac{1}{2L_3} |c(X_s, \mathcal{L}_{X_s})|^2 \int_{\mathbb{R}_0^d} |z| \left((1 + L_0|z|)^{p-1} - 1 \right) \nu(dz) \Bigg] ds$$
$$+ \int_0^t \int_{\mathbb{R}_0^d} e^{-\lambda s} \sum_{i=2}^{p/2} \sum_{j=0}^{i} \binom{p/2}{i} |z|^{2i-j} Q_p \left(p-2, 2i-j, |X_s|, 1 + \sqrt{\mathbb{E}[|X_s|^2]} \right) \nu(dz) ds$$
$$+ p \int_0^t e^{-\lambda s} |X_s|^{p-2} \langle X_s, \sigma(X_s, \mathcal{L}_{X_s}) dW_s \rangle$$

$$+\int_0^t \int_{\mathbb{R}_0^d} e^{-\lambda s}\left(\left|X_{s-}+c\left(X_{s-},\mathcal{L}_{X_{s-}}\right)z\right|^p-|X_{s-}|^p\right)\widetilde{N}(ds,dz).$$

Now, we define $\tau_N := \inf\{t \geq 0 : |X_t| \geq N\}$ for each $N > 0$. Choosing $\lambda = \gamma_1 p$ and using Condition **C7**, Remark 3.2.2 together with $\mathcal{W}_2^2(\mathcal{L}_{X_s},\delta_0) = \mathbb{E}[|X_s|^2]$, we obtain that

$$\begin{aligned}
&\mathbb{E}\left[e^{-\gamma_1 p(t\wedge\tau_N)}|X_{t\wedge\tau_N}|^p\right]\\
&\leq |x_0|^p + p\int_0^t e^{-\gamma_1 ps}\mathbb{E}\left[|X_s|^{p-2}\right]\left(\gamma_2\mathcal{W}_2^2(\mathcal{L}_{X_s},\delta_0)+\eta\right)ds + \int_0^t\int_{\mathbb{R}_0^d} e^{-\gamma_1 ps}\\
&\quad\times\sum_{i=2}^{p/2}\sum_{j=0}^{i}\binom{p/2}{i}|z|^{2i-j}\mathbb{E}\left[Q_p\left(p-2,2i-j,|X_s|,1+\sqrt{\mathbb{E}[|X_s|^2]}\right)\right]\nu(dz)ds\\
&\leq |x_0|^p + p\int_0^t e^{-\gamma_1 ps}\mathbb{E}\left[|X_s|^{p-2}\right]\left(\gamma_2\mathbb{E}\left[|X_s|^2\right]+\eta\right)ds + \int_0^t\int_{\mathbb{R}_0^d} e^{-\gamma_1 ps}\\
&\quad\times\sum_{i=2}^{p/2}\sum_{j=0}^{i}\binom{p/2}{i}|z|^{2i-j}\mathbb{E}\left[Q_p\left(p-2,2i-j,|X_s|,1+\sqrt{\mathbb{E}[|X_s|^2]}\right)\right]\nu(dz)ds.
\end{aligned} \tag{3.10}$$

Next, using $e^{-\gamma_1 p(t\wedge\tau_N)} \geq e^{-\gamma_1 pt}$ and the induction assumption (3.5), there exists a positive constant $C(T,p)$ which does not depend on N such that

$$\sup_{t\in[0,T]}\mathbb{E}\left[|X_{t\wedge\tau_N}|^p\right] \leq C(T,p). \tag{3.11}$$

This yields to

$$\sup_{t\in[0,T]}\mathbb{P}(\tau_N < t) \leq \frac{C(T,p)}{N^p},$$

which implies that τ_N tends to infinity a.s. as N tends to infinity. Now, it suffices to let $N \uparrow \infty$ and use Fatou's lemma for the left hand side of (3.11) in order to get that $\sup_{t\in[0,T]}\mathbb{E}[|X_t|^p] \leq C(T,p)$. Thus, by the induction principle, we have shown (3.4).

Step 2: We next wish to show (3.3) for any even natural number $p \in [2, p_0]$. First, applying (3.6) to $\lambda = 2\gamma$ and $p = 2$, and using **C7** and $\mathcal{W}_2^2(\mathcal{L}_{X_s},\delta_0) = \mathbb{E}[|X_s|^2]$, we get

$$\begin{aligned}
&e^{-2\gamma t}|X_t|^2\\
&= |x_0|^2 + \int_0^t e^{-2\gamma s}\left[-2\gamma|X_s|^2 + 2\langle X_s, b(X_s,\mathcal{L}_{X_s})\rangle + |\sigma(X_s,\mathcal{L}_{X_s})|^2\right]ds
\end{aligned}$$

$$
\begin{aligned}
&+2\int_0^t e^{-2\gamma s}\left\langle X_s,\sigma\left(X_s,\mathcal{L}_{X_s}\right)dW_s\right\rangle+\int_0^t\int_{\mathbb{R}_0^d}e^{-2\gamma s}\Big[\left|X_{s-}+c\left(X_{s-},\mathcal{L}_{X_{s-}}\right)z\right|^2\\
&-\left|X_{s-}\right|^2\Big]\widetilde{N}(ds,dz)+\int_0^t\int_{\mathbb{R}_0^d}e^{-2\gamma s}|c\left(X_s,\mathcal{L}_{X_s}\right)z|^2\nu(dz)ds\\
\leq{}&|x_0|^2+2\int_0^t e^{-2\gamma s}\Bigg[-\gamma\left|X_s\right|^2+\left\langle X_s,b\left(X_s,\mathcal{L}_{X_s}\right)\right\rangle+\frac{1}{2}\left|\sigma\left(X_s,\mathcal{L}_{X_s}\right)\right|^2\\
&+\frac{1}{2}|c\left(X_s,\mathcal{L}_{X_s}\right)|^2\int_{\mathbb{R}_0^d}|z|^2\nu(dz)\Bigg]ds+2\int_0^t e^{-2\gamma s}\left\langle X_s,\sigma\left(X_s,\mathcal{L}_{X_s}\right)dW_s\right\rangle\\
&+\int_0^t\int_{\mathbb{R}_0^d}e^{-2\gamma s}\left(\left|X_{s-}+c\left(X_{s-},\mathcal{L}_{X_{s-}}\right)z\right|^2-\left|X_{s-}\right|^2\right)\widetilde{N}(ds,dz)\\
\leq{}&|x_0|^2+2\int_0^t e^{-2\gamma s}\left(-\gamma_2\left|X_s\right|^2+\gamma_2\mathcal{W}_2^2\left(\mathcal{L}_{X_s},\delta_0\right)+\eta\right)ds\\
&+2\int_0^t e^{-2\gamma s}\langle X_s,\sigma\left(X_s,\mathcal{L}_{X_s}\right)dW_s\rangle\\
&+\int_0^t\int_{\mathbb{R}_0^d}e^{-2\gamma s}\left(\left|X_{s-}+c\left(X_{s-},\mathcal{L}_{X_{s-}}\right)z\right|^2-\left|X_{s-}\right|^2\right)\widetilde{N}(ds,dz). \qquad (3.12)
\end{aligned}
$$

Thanks to the fact that $\mathcal{W}_2^2\left(\mathcal{L}_{X_s},\delta_0\right)=\mathbb{E}[|X_s|^2]$, and the estimate (3.4), we get

$$
\mathbb{E}\left[e^{-2\gamma t}|X_t|^2\right]\leq|x_0|^2+2\eta\int_0^t e^{-2\gamma s}ds,
$$

which yields to

$$
\mathbb{E}\left[|X_t|^2\right]\leq\begin{cases}\left(|x_0|^2+\dfrac{\eta}{\gamma}\right)e^{2\gamma t}-\dfrac{\eta}{\gamma} & \text{if}\quad\gamma\neq 0,\\ |x_0|^2+2\eta t & \text{if}\quad\gamma=0.\end{cases}
$$

Thus, (3.3) holds for $p=2$.

Now, we suppose that (3.3) is valid for all even integer $q\in[2,p-2]$. That is,

$$
\mathbb{E}\left[|X_t|^q\right]\leq\begin{cases}C_q(1+e^{\gamma qt}) & \text{if}\quad\gamma\neq 0,\\ C_q(1+t)^{q/2} & \text{if}\quad\gamma=0,q=2\ \text{ or }\ \gamma=0,\gamma_2>0,q\in(2,p-2],\\ C_q(1+t)^q & \text{if}\quad\gamma=0,\gamma_2=0,q\in(2,p-2].\end{cases} \qquad (3.13)
$$

We are going to show that (3.3) holds for even integer p. For this, it suffices to use (3.10), the inductive assumption (3.13) and Condition **C5**.

<u>Case $\gamma\neq 0$:</u> We have

$$
\mathbb{E}\left[e^{-\gamma_1 p(t\wedge\tau_N)}|X_{t\wedge\tau_N}|^p\right]\leq|x_0|^p+C\int_0^t e^{-\gamma_1 ps}\left(1+e^{\gamma pt}\right)ds
$$

$$= |x_0|^p + C\int_0^t \left(e^{-\gamma_1 ps} + e^{\gamma_2 ps}\right) ds.$$

Thanks to fact that $\tau_N \uparrow \infty$ a.s. as $N \uparrow \infty$, applying Fatou's lemma, we get

$$\mathbb{E}\left[e^{-\gamma_1 pt}|X_t|^p\right] \leq |x_0|^p + C\int_0^t \left(e^{-\gamma_1 ps} + e^{\gamma_2 ps}\right) ds.$$

When $\gamma_1 = 0$, we have

$$\begin{aligned}\mathbb{E}\left[|X_t|^p\right] &\leq |x_0|^p + C\left(1 + t + e^{\gamma_2 pt}\right)\\ &\leq C\left(1 + e^{\gamma pt}\right).\end{aligned}$$

When $\gamma_1 \neq 0$, we have

$$\begin{aligned}\mathbb{E}\left[|X_t|^p\right] &\leq |x_0|^p e^{\gamma_1 pt} + \frac{C}{p\gamma_1}\left(e^{\gamma_1 pt} - 1\right) + \frac{C}{p\gamma_2}e^{\gamma pt}\\ &\leq C\left(1 + e^{\gamma pt}\right).\end{aligned}$$

<u>Case $\gamma = 0$:</u> When $\gamma_2 > 0$, we have

$$\begin{aligned}\mathbb{E}\left[e^{-\gamma_1 p(t\wedge\tau_N)}|X_{t\wedge\tau_N}|^p\right] &\leq |x_0|^p + C\int_0^t e^{-\gamma_1 ps}\left(1+s\right)^{p/2} ds\\ &\leq |x_0|^p + C\left(1+t\right)^{p/2}\int_0^t e^{-\gamma_1 ps} ds.\end{aligned}$$

Then, letting $N \uparrow \infty$ and using $-\gamma_1 = \gamma_2$, we obtain

$$\mathbb{E}\left[e^{\gamma_2 pt}|X_t|^p\right] \leq |x_0|^p + C\left(1+t\right)^{p/2}\int_0^t e^{\gamma_2 ps} ds \leq |x_0|^p + \frac{C}{\gamma_2 p}\left(1+t\right)^{p/2} e^{\gamma_2 pt}.$$

Hence,

$$\mathbb{E}\left[|X_t|^p\right] \leq C\left(1+t\right)^{p/2}.$$

When $\gamma_2 = \gamma_1 = 0$, we have

$$\mathbb{E}\left[|X_{t\wedge\tau_N}|^p\right] \leq |x_0|^p + C\int_0^t \left(1+s\right)^{p-1} ds \leq C\left(1+t\right)^p.$$

Then, letting $N \uparrow \infty$, we obtain

$$\mathbb{E}\left[|X_t|^p\right] \leq C\left(1+t\right)^p.$$

Consequently, (3.3) holds for even integer p. Due to the induction principle, (3.3) is valid for any even natural number $p \in [2, p_0]$. Finally, (3.3) is also valid for any $p \in [2, p_0]$ thanks to Hölder's inequality. This finishes the proof. □

3.4 Propagation of chaos

McKean-Vlasov SDEs have garnered renewed attention in recent years, largely due to the resolution of certain technical hurdles, but also owing to their utility in modeling across diverse fields, including statistical physics, neuroscience, and finance, among others. At the core of this interest is the fact that McKean-Vlasov SDEs describe a limiting behavior of individual particles interacting with each other in a "mean-field" manner — the well-known Propagation of Chaos result. The dependence of the coefficients on $\mathcal{L}_{X(t)}$ signifies that the behavior of each particle (represented by $X(t)$) is influenced not only by external factors (noise $W(t)$ and $N(t)$) and its own present state, but also by the average state of the entire system (reflected through the distribution $\mathcal{L}_{X(t)}$. As the number of particles approaches infinity, rather than tracking each individual particle (a computationally expensive task), one can simply describe the behavior of a "representative" particle and how it interacts with the "mean-field" generated by all the other particles. This latter result also paves the way, among other things, for the numerical solution of such McKean-Vlasov SDEs through approximation of the associated system of interacting particles.

For $N \in \mathbb{N}$, suppose that (W^i, Z^i) are independent copies of the couple (W, Z) for $i \in \{1, \ldots, N\}$. Let $N^i(dt, dz)$ be the Poisson random measure associated with the jumps of the Lévy process Z^i with intensity measure $\nu(dz)dt$, and $\widetilde{N}^i(dt, dz) := N^i(dt, dz) - \nu(dz)dt$ be the compensated Poisson random measure associated with $N^i(dt, dz)$. Thus, the Lévy-Itô decomposition of Z^i is given by $Z^i_t = \int_0^t \int_{\mathbb{R}^d_0} z\widetilde{N}^i(ds, dz)$ for $t \geq 0$. We now consider the system of non-interacting particles, which is associated with the Lévy-driven McKean-Vlasov SDE (3.1), where the state $X^i = (X^i_t)_{t\geq 0}$ of particle i is defined by

$$\begin{aligned} X^i_t &= x_0 + \int_0^t b(X^i_s, \mathcal{L}_{X^i_s})ds + \int_0^t \sigma(X^i_s, \mathcal{L}_{X^i_s})dW^i_s + \int_0^t c\left(X^i_{s-}, \mathcal{L}_{X^i_{s-}}\right) dZ^i_s \\ &= x_0 + \int_0^t b(X^i_s, \mathcal{L}_{X^i_s})ds + \int_0^t \sigma(X^i_s, \mathcal{L}_{X^i_s})dW^i_s \\ &\quad + \int_0^t \int_{\mathbb{R}^d_0} c\left(X^i_{s-}, \mathcal{L}_{X^i_{s-}}\right) z\widetilde{N}^i(ds, dz), \end{aligned} \tag{3.14}$$

for any $t \geq 0$ and $i \in \{1, \ldots, N\}$.

For $\boldsymbol{x}^N := (x_1, x_2, \dots, x_N), \boldsymbol{y}^N := (y_1, y_2, \dots, y_N) \in \mathbb{R}^{dN}$, we have

$$\mathcal{W}_2^2(\mu^{\boldsymbol{x}^N}, \delta_0) = \frac{1}{N}\sum_{i=1}^N |x_i|^2 .$$

Here, the empirical measure is defined by $\mu^{\boldsymbol{x}^N}(dx) := \frac{1}{N}\sum_{i=1}^N \delta_{x_i}(dx)$, where δ_x denotes the Dirac measure at x. Moreover, a standard bound for the Wasserstein distance between two empirical measures $\mu^{\boldsymbol{x}^N}, \mu^{\boldsymbol{y}^N}$ is given by

$$\mathcal{W}_2^2(\mu^{\boldsymbol{x}^N}, \mu^{\boldsymbol{y}^N}) \leq \frac{1}{N}\sum_{i=1}^N |x_i - y_i|^2 = \frac{1}{N}\left|\boldsymbol{x}^N - \boldsymbol{y}^N\right|^2 ,$$

(see (1.24) of [6]).

Now, the true measure $\mathcal{L}_{X_t}$ at each time t is approximated by the empirical measure

$$\mu_t^{\boldsymbol{X}^N}(dx) := \frac{1}{N}\sum_{i=1}^N \delta_{X_t^{i,N}}(dx), \tag{3.15}$$

where $\boldsymbol{X}^N = (\boldsymbol{X}_t^N)_{t\geq 0} = (X_t^{1,N}, \dots, X_t^{N,N})_{t\geq 0}^{\mathsf{T}}$, which is called the system of interacting particles, is the solution to the $\mathbb{R}^{dN}$-dimensional Lévy-driven SDE with components $X^{i,N} = (X_t^{i,N})_{t\geq 0}$

$$\begin{aligned} X_t^{i,N} &= x_0 + \int_0^t b(X_s^{i,N}, \mu_s^{\boldsymbol{X}^N})ds + \int_0^t \sigma(X_s^{i,N}, \mu_s^{\boldsymbol{X}^N})dW_s^i + \int_0^t c\left(X_{s-}^{i,N}, \mu_{s-}^{\boldsymbol{X}^N}\right) dZ_s^i \\ &= x_0 + \int_0^t b(X_s^{i,N}, \mu_s^{\boldsymbol{X}^N})ds + \int_0^t \sigma(X_s^{i,N}, \mu_s^{\boldsymbol{X}^N})dW_s^i \\ &\quad + \int_0^t \int_{\mathbb{R}_0^d} c\left(X_{s-}^{i,N}, \mu_{s-}^{\boldsymbol{X}^N}\right) z\widetilde{N}^i(ds, dz), \end{aligned} \tag{3.16}$$

for any $t \geq 0$ and $i \in \{1, \dots, N\}$.

Observe that the interacting particle system $\boldsymbol{X}^N = (X^{i,N})_{i\in\{1,\dots,N\}}^{\mathsf{T}}$ can be viewed as an ordinary Lévy-driven SDE with random coefficients taking values in $\mathbb{R}^{dN}$. Therefore, under Conditions **C1**, **C3** and **C2** valid for $\kappa_1 = \kappa_2 = 1, L_1 = L_2 > 0$, there exists a unique càdlàg solution such that

$$\max_{i\in\{1,\dots,N\}} \sup_{t\in[0,T]} \mathbb{E}\left[|X_t^{i,N}|^2\right] \leq K,$$

for any $N \in \mathbb{N}$, where $K > 0$ does not depend on N.

Proposition 3.4.1. *Let $X^{i,N} = (X_t^{i,N})_{t\geq 0}$ be a solution to equation (3.16). Assume Conditions **C6, C7** hold and that σ is bounded on $C \times \mathcal{P}_2(\mathbb{R}^d)$ for every compact subset*

*C of $\mathbb{R}^d$, and **C5** holds for $q = 2p_0$. Then for any $p \in [2; p_0]$, there exists a positive constant C_p such that for any $t \geq 0$,*

$$\max_{i\in\{1,\dots,N\}} \mathbb{E}\left[|X_t^{i,N}|^p\right] \leq \begin{cases} C_p(1+e^{\gamma p t}) & \textit{if} \quad \gamma \neq 0, \\ C_p(1+t)^{p/2} & \textit{if} \quad \gamma = 0, p = 2 \quad \textit{or} \quad \gamma = 0, \gamma_2 > 0, p \in (2, p_0], \\ C_p(1+t)^p & \textit{if} \quad \gamma = 0, \gamma_2 = 0, p \in (2, p_0], \end{cases}$$

where $\gamma = \gamma_1 + \gamma_2$.

Note that when $\gamma < 0$, we have that $\max_{i\in\{1,\dots,N\}} \sup_{t\geq 0} \mathbb{E}\left[|X_t^{i,N}|^p\right] \leq 2C_p$.

Proof. The proof follows the same lines as the one of Proposition 3.3.2, thus we omit it. □

Finally in this section, we provide a result on the propagation of chaos which is the key to the convergence as $N \uparrow \infty$. The new achievement of our research, in contrast to previous studies, lies in establishing the convergence rate of the system of N interacting particles towards the mean-field limit, described by the McKean-Vlasov stochastic differential equation, over infinite time intervals. This result is articulated in the estimate (3.18) of Proposition 3.4.2.

To simplify the exposition, we define

$$\varphi(N) = \begin{cases} N^{-1/2} & \text{if } \ d < 4, \\ N^{-1/2} \ln N & \text{if } \ d = 4, \\ N^{-2/d} & \text{if } \ d > 4. \end{cases}$$

Proposition 3.4.2. *Assume that all conditions in Proposition 3.4.1 hold and that Condition **C2** holds for $\kappa_1 = \kappa_2 = 1$, $L_1 \in \mathbb{R}$, $L_2 \geq 0$. Then, we have*

$$\max_{i\in\{1,\dots,N\}} \sup_{t\in[0,T]} \mathbb{E}\left[\left|X_t^i - X_t^{i,N}\right|^2\right] \leq C_T \varphi(N), \tag{3.17}$$

for any $N \in \mathbb{N}$, where the positive constant C_T does not depend on N.

Assume further that $L_1 + L_2 < 0$ and $\gamma < 0$. Then, we have

$$\max_{i\in\{1,\dots,N\}} \sup_{t\geq 0} \mathbb{E}\left[\left|X_t^i - X_t^{i,N}\right|^2\right] \leq C\varphi(N), \tag{3.18}$$

for any $N \in \mathbb{N}$, where the positive constant C does not depend on N and T.

Proof. Observe that for any $t \geq 0$,

$$X_t^i - X_t^{i,N} = \int_0^t \Big(b(X_s^i, \mathcal{L}_{X_s^i}) - b(X_s^{i,N}, \mu_s^{\boldsymbol{X}^N}) \Big)\, ds + \int_0^t \Big(\sigma(X_s^i, \mathcal{L}_{X_s^i}) - \sigma(X_s^{i,N}, \mu_s^{\boldsymbol{X}^N}) \Big)\, dW_s^i + \int_0^t \int_{\mathbb{R}_0^d} \Big(c(X_{s-}^i, \mathcal{L}_{X_{s-}^i}) - c(X_{s-}^{i,N}, \mu_{s-}^{\boldsymbol{X}^N}) \Big)\, z\widetilde{N}^i(ds, dz).$$

Then, for $\lambda \in \mathbb{R}$, applying Itô's formula and Condition **C2** valid for $\kappa_1 = \kappa_2 = 1$, $L_1 \in \mathbb{R}$, $L_2 \geq 0$, we obtain that for any $t \geq 0$,

$$\begin{aligned}
& e^{-\lambda t}|X_t^i - X_t^{i,N}|^2 \\
&= \int_0^t e^{-\lambda s}\Big[-\lambda|X_s^i - X_s^{i,N}|^2 + 2\langle X_s^i - X_s^{i,N}, b(X_s^i, \mathcal{L}_{X_s^i}) - b(X_s^{i,N}, \mu_s^{\boldsymbol{X}^N})\rangle \\
&\quad + |\sigma(X_s^i, \mathcal{L}_{X_s^i}) - \sigma(X_s^{i,N}, \mu_s^{\boldsymbol{X}^N})|^2\Big] ds \\
&\quad + 2\int_0^t e^{-\lambda s}\langle X_s^i - X_s^{i,N}, (\sigma(X_s^i, \mathcal{L}_{X_s^i}) - \sigma(X_s^{i,N}, \mu_s^{\boldsymbol{X}^N}))dW_s^i\rangle \\
&\quad + \int_0^t \int_{\mathbb{R}_0^d} e^{-\lambda s}\Big[|X_{s-}^i - X_{s-}^{i,N} + \big(c(X_{s-}^i, \mathcal{L}_{X_{s-}^i}) - c(X_{s-}^{i,N}, \mu_{s-}^{\boldsymbol{X}^N})\big)z|^2 - |X_{s-}^i - X_{s-}^{i,N}|^2\Big] \\
&\quad \times \widetilde{N}^i(ds, dz) + \int_0^t \int_{\mathbb{R}_0^d} e^{-\lambda s}\Big|\big(c(X_{s-}^i, \mathcal{L}_{X_{s-}^i}) - c(X_{s-}^{i,N}, \mu_{s-}^{\boldsymbol{X}^N})\big)z\Big|^2 \nu(dz)ds \\
&\leq \int_0^t e^{-\lambda s}\Bigg[-\lambda|X_s^i - X_s^{i,N}|^2 + 2\langle X_s^i - X_s^{i,N}, b(X_s^i, \mathcal{L}_{X_s^i}) - b(X_s^{i,N}, \mu_s^{\boldsymbol{X}^N})\rangle \\
&\quad + |\sigma(X_s^i, \mathcal{L}_{X_s^i}) - \sigma(X_s^{i,N}, \mu_s^{\boldsymbol{X}^N})|^2 + |c(X_{s-}^i, \mathcal{L}_{X_{s-}^i}) - c(X_{s-}^{i,N}, \mu_{s-}^{\boldsymbol{X}^N}|^2 \int_{\mathbb{R}_0^d} |z|^2\nu(dz)\Bigg] ds \\
&\quad + 2\int_0^t e^{-\lambda s}\langle X_s^i - X_s^{i,N}, (\sigma(X_s^i, \mathcal{L}_{X_s^i}) - \sigma(X_s^{i,N}, \mu_s^{\boldsymbol{X}^N}))dW_s^i\rangle \\
&\quad + \int_0^t \int_{\mathbb{R}_0^d} e^{-\lambda s}\Big[\Big|X_{s-}^i - X_{s-}^{i,N} + \big(c(X_{s-}^i, \mathcal{L}_{X_{s-}^i}) - c(X_{s-}^{i,N}, \mu_{s-}^{\boldsymbol{X}^N})\big)z\Big|^2 \\
&\quad - |X_{s-}^i - X_{s-}^{i,N}|^2\Big] \widetilde{N}^i(ds, dz) \\
&\leq \int_0^t e^{-\lambda s}\Big(-\lambda|X_s^i - X_s^{i,N}|^2 + L_1|X_s^i - X_s^{i,N}|^2 + L_2\mathcal{W}_2^2(\mathcal{L}_{X_s^i}, \mu_s^{\boldsymbol{X}^N})\Big) ds \\
&\quad + 2\int_0^t e^{-\lambda s}\langle X_s^i - X_s^{i,N}, (\sigma(X_s^i, \mathcal{L}_{X_s^i}) - \sigma(X_s^{i,N}, \mu_s^{\boldsymbol{X}^N}))dW_s^i\rangle \\
&\quad + \int_0^t \int_{\mathbb{R}_0^d} e^{-\lambda s}\Big(\big|X_{s-}^i - X_{s-}^{i,N} + \big(c(X_{s-}^i, \mathcal{L}_{X_{s-}^i}) - c(X_{s-}^{i,N}, \mu_{s-}^{\boldsymbol{X}^N})\big)z|^2
\end{aligned}$$

$$- |X^i_{s-} - X^{i,N}_{s-}|^2\Big) \widetilde{N}^i(ds,dz).$$

Therefore, taking the expectation and using the estimate $\mathcal{W}_2^2(\mathcal{L}_{X^i_s}, \mathcal{L}_{X^{i,N}_s}) \leq \mathbb{E}[|X^i_s - X^{i,N}_s|^2]$, we obtain that for any $\varepsilon > 0$,

$$\begin{aligned}
& e^{-\lambda t}\mathbb{E}\left[\left|X^i_t - X^{i,N}_t\right|^2\right] \\
&\leq \int_0^t e^{-\lambda s}\Big((-\lambda + L_1)\mathbb{E}\left[|X^i_s - X^{i,N}_s|^2\right] + L_2\mathbb{E}\left[\mathcal{W}_2^2(\mathcal{L}_{X^i_s}, \mu^{\boldsymbol{X}^N}_s)\right]\Big)\, ds \\
&\leq \int_0^t e^{-\lambda s}\Bigg[(-\lambda + L_1)\mathbb{E}\left[|X^i_s - X^{i,N}_s|^2\right] \\
&\quad + L_2\Big(\big(1+\frac{\varepsilon}{L_2}\big)\mathbb{E}\left[\mathcal{W}_2^2(\mathcal{L}_{X^i_s}, \mathcal{L}_{X^{i,N}_s})\right] + \big(1+\frac{L_2}{\varepsilon}\big)\mathbb{E}\left[\mathcal{W}_2^2(\mathcal{L}_{X^{i,N}_s}, \mu^{\boldsymbol{X}^N}_s)\right]\Big)\Bigg] ds \\
&\leq \int_0^t e^{-\lambda s}\Bigg[(-\lambda + L_1 + L_2 + \varepsilon)\,\mathbb{E}\left[|X^i_s - X^{i,N}_s|^2\right] \\
&\quad + L_2\big(1+\frac{L_2}{\varepsilon}\big)\mathbb{E}\left[\mathcal{W}_2^2(\mathcal{L}_{X^{i,N}_s}, \mu^{\boldsymbol{X}^N}_s)\right]\Bigg] ds.
\end{aligned} \tag{3.19}$$

Moreover, from Proposition 3.4.1, we have that for any $p \in (4, p_0]$,

$$\max_{i\in\{1,\ldots,N\}} \sup_{t\in[0,T]} \mathbb{E}\left[\left|X^{i,N}_t\right|^p\right] \leq C_T,$$

for some constant $C_T > 0$. This, together with [7, Theorem 5.8], deduces that

$$\begin{aligned}
\max_{i\in\{1,\ldots,N\}} \mathbb{E}\left[\mathcal{W}_2^2(\mathcal{L}_{X^{i,N}_s}, \mu^{\boldsymbol{X}^N}_s)\right] &\leq C\begin{cases} N^{-1/2} & \text{if } d < 4, \\ N^{-1/2}\ln N & \text{if } d = 4, \\ N^{-2/d} & \text{if } d > 4 \end{cases} \\
&= C\varphi(N),
\end{aligned} \tag{3.20}$$

for any $s \in [0,T]$, where the positive constant C does not depend on the time.

Consequently, it suffices to choose $\lambda = L_1 + L_2 + \varepsilon$ in (3.19) and use the estimate (3.20) in order to conclude (3.17).

Finally, when $\gamma < 0$, it follows from Proposition 3.4.1 that for any $p \in (4, p_0]$,

$$\max_{i\in\{1,\ldots,N\}} \sup_{t\geq 0} \mathbb{E}\left[\left|X^{i,N}_t\right|^q\right] \leq C,$$

where the positive constant C does not depend on T. Furthermore, when $L_1 + L_2 < 0$, one can always choose ε sufficiently small such that $\lambda < 0$. This allows us to conclude (3.18), and the proof is complete. □

3.5 Tamed-adaptive Euler-Maruyama scheme

In this section, we present a novel Euler-Maruyama approximation scheme for equation (3.1). Approximating solutions for McKean-Vlasov stochastic differential equations poses greater challenges than for standard stochastic differential equations. Our approach involves a two-step process. First, we approximate the measure $\mathcal{L}_{X_t}$, as detailed in Section 3.4. Second, we approximate each process $X_t^{i,N}$ within the system (3.16) using the tamed-adaptive Euler-Maruyama scheme previously described in Chapter 2.

Let $\sigma_\Delta = (\sigma_{\Delta,ij})_{1\le i,j\le d} : \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}^d \otimes \mathbb{R}^d$ and $c_\Delta = (c_{\Delta,ij})_{1\le i,j\le d} : \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}^d \otimes \mathbb{R}^d$ be approximations of the coefficients σ and c, respectively, which will be specified later. For all $i \in \{1, \dots, N\}$, $\Delta \in (0, 1)$ and $k \in \mathbb{N}$, we define the tamed-adaptive Euler-Maruyama discretization of equation (3.16) by

$$\begin{cases} t_0 = 0, \quad \widehat{X}_0^{i,N} = x_0, \quad t_{k+1} = t_k + \mathbf{h}(\widehat{\boldsymbol{X}}_{t_k}^N, \mu_{t_k}^{\widehat{\boldsymbol{X}}^N})\Delta, \\ \widehat{X}_{t_{k+1}}^{i,N} = \widehat{X}_{t_k}^{i,N} + b(\widehat{X}_{t_k}^{i,N}, \mu_{t_k}^{\widehat{\boldsymbol{X}}^N})(t_{k+1} - t_k) + \sigma_\Delta(\widehat{X}_{t_k}^{i,N}, \mu_{t_k}^{\widehat{\boldsymbol{X}}^N})(W_{t_{k+1}}^i - W_{t_k}^i) \\ \qquad + c_\Delta(\widehat{X}_{t_k}^{i,N}, \mu_{t_k}^{\widehat{\boldsymbol{X}}^N})(Z_{t_{k+1}}^i - Z_{t_k}^i), \end{cases} \tag{3.21}$$

where

$$\widehat{\boldsymbol{X}}_{t_k}^N = \left(\widehat{X}_{t_k}^{1,N}, \dots, \widehat{X}_{t_k}^{N,N}\right),$$

$$\mu_{t_k}^{\widehat{\boldsymbol{X}}^N}(dx) := \frac{1}{N}\sum_{i=1}^N \delta_{\widehat{X}_{t_k}^{i,N}}(dx),$$

$$\mathbf{h}(\widehat{\boldsymbol{X}}_{t_k}^N, \mu_{t_k}^{\widehat{\boldsymbol{X}}^N}) = \min\left\{h(\widehat{X}_{t_k}^{1,N}, \mu_{t_k}^{\widehat{\boldsymbol{X}}^N}), \dots, h(\widehat{X}_{t_k}^{N,N}, \mu_{t_k}^{\widehat{\boldsymbol{X}}^N})\right\},$$

and

$$h(x,\mu) = \frac{h_0}{(1 + |b(x,\mu)| + |\sigma(x,\mu)| + |x|^\ell)^2 + |c(x,\mu)|^{p_0}}, \tag{3.22}$$

for $x \in \mathbb{R}^d$, $\mu \in \mathcal{P}_2(\mathbb{R}^d)$ and some positive constant h_0. Here, the constants ℓ and p_0 are respectively defined in Conditions **C4** and **C7**.

In all what follows, to simplify the exposition, we take $h_0 = 1$ in the proofs.

Analogous to the approximation schemes presented in Chapter 2, we must provide sufficient conditions to guarantee that $t_k \uparrow \infty$ as $k \uparrow \infty$, thereby demonstrating that the tamed-adaptive Euler-Maruyama approximation scheme (3.21) is well-defined.

Proposition 3.5.1. *Assume that Condition **C5** holds for $p = 2$ and there exist positive constants L, β_1 and β_2 such that the functions h, b, σ_Δ and c_Δ satisfy*

T1. $\dfrac{1}{h(x,\mu)} \leq L\left(1+|x|^{\beta_1}+\mathcal{W}_2^{\beta_2}(\mu,\delta_0)\right)$; $|b(x,\mu)|\left(1+|b(x,\mu)|\right)h(x,\mu) \leq L$;

T2. $\langle x, b(x,\mu)-b(0,\delta_0)\rangle \leq L\left(|x|^2+\mathcal{W}_2^2(\mu,\delta_0)\right)$;

T3. $|\sigma_\Delta(x,\mu)|\left(1+|x|\right) \leq \dfrac{L}{\sqrt{\Delta}}$; $|c_\Delta(x,\mu)|\left(1+|x|\right) \leq \dfrac{L}{\sqrt{\Delta}}$;
$|b(x,\mu)||c_\Delta(x,\mu)| \leq \dfrac{L}{\sqrt{\Delta}}$;

for any $x \in \mathbb{R}^d$ and $\mu \in \mathcal{P}_2(\mathbb{R}^d)$. Then, we have

$$\lim_{k\to+\infty} t_k = +\infty \quad a.s.$$

Proof. For all $i \in \{1,\dots,N\}$ and $H>0$, we define the tamed-adaptive Euler-Maruyama discretization of equation (3.16) as follows

$$\begin{cases} t_0 = 0, \quad \widehat{X}_0^{i,N,H} = x_0, \quad t_{k+1}^H = t_k^H + \mathbf{h}^H(\widehat{\boldsymbol{X}}_{t_k^H}^{N,H}, \mu_{t_k^H}^{\widehat{\boldsymbol{X}}^{N,H}})\Delta, \\ \widehat{X}_{t_{k+1}^H}^{i,N,H} = \widehat{X}_{t_k^H}^{i,N,H} + b_H(\widehat{X}_{t_k^H}^{i,N,H}, \mu_{t_k^H}^{\widehat{\boldsymbol{X}}^{N,H}})(t_{k+1}^H - t_k^H) \\ \qquad +\sigma_\Delta(\widehat{X}_{t_k^H}^{i,N,H}, \mu_{t_k^H}^{\widehat{\boldsymbol{X}}^{N,H}})(W_{t_{k+1}^H}^i - W_{t_k^H}^i) + c_\Delta(\widehat{X}_{t_k^H}^{i,N,H}, \mu_{t_k^H}^{\widehat{\boldsymbol{X}}^{N,H}})(Z_{t_{k+1}^H}^i - Z_{t_k^H}^i), \end{cases} \tag{3.23}$$

where

$$\widehat{\boldsymbol{X}}_{t_k}^{N,H} = \left(\widehat{X}_{t_k}^{1,N,H},\dots,\widehat{X}_{t_k}^{N,N,H}\right),$$

$$\mu_{t_k}^{\widehat{\boldsymbol{X}}^{N,H}}(dx) := \frac{1}{N}\sum_{i=1}^N \delta_{\widehat{X}_{t_k}^{i,N,H}}(dx),$$

$$\mathbf{h}^H(\widehat{\boldsymbol{X}}_{t_k^H}^N, \mu_{t_k^H}^{\widehat{\boldsymbol{X}}^{N,H}}) = \min\left\{h^H(\widehat{X}_{t_k^H}^{1,N,H}, \mu_{t_k^H}^{\widehat{\boldsymbol{X}}^{N,H}}),\dots,h^H(\widehat{X}_{t_k^H}^{N,N,H}, \mu_{t_k^H}^{\widehat{\boldsymbol{X}}^{N,H}})\right\},$$

$$h^H(x,\mu) = \begin{cases} h(x,\mu) & \text{if} \quad |x|^{\beta_1}+\mathcal{W}_2^{\beta_2}(\mu,\delta_0) \leq H, \\ \dfrac{1}{1+H} & \text{if} \quad |x|^{\beta_1}+\mathcal{W}_2^{\beta_2}(\mu,\delta_0) > H, \end{cases}$$

$$b_H(x,\mu) = \begin{cases} b(x,\mu) & \text{if} \quad |x|^{\beta_1}+\mathcal{W}_2^{\beta_2}(\mu,\delta_0) \leq H, \\ \dfrac{x}{1+|x|^2} + b(0,\delta_0) & \text{if} \quad |x|^{\beta_1}+\mathcal{W}_2^{\beta_2}(\mu,\delta_0) > H, \end{cases}$$

for $x \in \mathbb{R}^d$ and $\mu \in \mathcal{P}_2(\mathbb{R}^d)$. Then, it can be checked that for all $x \in \mathbb{R}^d$ and $\mu \in \mathcal{P}_2(\mathbb{R}^d)$,

(1) $|b_H(x,\mu)|h^H(x,\mu) \leq C_0$ and $|b_H(x,\mu)|^2 h^H(x,\mu) \leq C_0$,

(2) $\langle x, b_H(x,\mu)-b(0,\delta_0)\rangle \leq C_0\left(|x|^2+\mathcal{W}_2^2(\mu,\delta_0)\right)$ due to Condition **T2**,

for some other positive constant C_0. Moreover, from Condition **T1**, we have

$$h^H(x,\mu)\Delta \geq \frac{\min\{1, L^{-1}\}\Delta}{1+H},$$

which implies that $t_{k+1}^H - t_k^H \geq \frac{\min\{1,L^{-1}\}\Delta}{1+H}$. Therefore,

$$\lim_{k\to+\infty} t_k^H = +\infty \quad \text{a.s.}$$

Now, we define by $\underline{t}^H := \max\left\{t_k^H : t_k^H \leq t\right\}$ the nearest time point before t. The continuous interpolant process is defined by

$$\begin{aligned}
\widehat{X}_t^{i,N,H} :&= \widehat{X}_{\underline{t}^H}^{i,N,H} + b_H\left(\widehat{X}_{\underline{t}^H}^{i,N,H}, \mu_{\underline{t}^H}^{\widehat{\boldsymbol{X}}^{N,H}}\right)(t-\underline{t}^H) + \sigma_\Delta\left(\widehat{X}_{\underline{t}^H}^{i,N,H}, \mu_{\underline{t}^H}^{\widehat{\boldsymbol{X}}^{N,H}}\right)\left(W_t^i - W_{\underline{t}^H}^i\right) \\
&\quad + c_\Delta\left(\widehat{X}_{\underline{t}^H}^{i,N,H}, \mu_{\underline{t}^H}^{\widehat{\boldsymbol{X}}^{N,H}}\right)\left(Z_t^i - Z_{\underline{t}^H}^i\right) \\
&= x_0 + \int_0^t b_H\left(\widehat{X}_{\underline{s}^H}^{i,N,H}, \mu_{\underline{s}^H}^{\widehat{\boldsymbol{X}}^{N,H}}\right)ds + \int_0^t \sigma_\Delta\left(\widehat{X}_{\underline{s}^H}^{i,N,H}, \mu_{\underline{s}^H}^{\widehat{\boldsymbol{X}}^{N,H}}\right)dW_s^i \\
&\quad + \int_0^t\int_{\mathbb{R}_0^d} c_\Delta\left(\widehat{X}_{\underline{s}^H}^{i,N,H}, \mu_{\underline{s}^H}^{\widehat{\boldsymbol{X}}^{N,H}}\right)z\widetilde{N}^i(ds,dz). \qquad (3.24)
\end{aligned}$$

Using Itô's formula, we get

$$\begin{aligned}
&|\widehat{X}_t^{i,N,H}|^2 \\
&= |x_0|^2 + \int_0^t \left[2\left\langle \widehat{X}_s^{i,N,H}, b_H\left(\widehat{X}_{\underline{s}^H}^{i,N,H}, \mu_{\underline{s}^H}^{\widehat{\boldsymbol{X}}^{N,H}}\right)\right\rangle + \left|\sigma_\Delta\left(\widehat{X}_{\underline{s}^H}^{i,N,H}, \mu_{\underline{s}^H}^{\widehat{\boldsymbol{X}}^{N,H}}\right)\right|^2\right]ds \\
&\quad + 2\int_0^t \left\langle \widehat{X}_s^{i,N,H}, \sigma_\Delta\left(\widehat{X}_{\underline{s}^H}^{i,N,H}, \mu_{\underline{s}^H}^{\widehat{\boldsymbol{X}}^{N,H}}\right)dW_s^i\right\rangle \\
&\quad + \int_0^t\int_{\mathbb{R}_0^d} \left|c_\Delta\left(\widehat{X}_{\underline{s}^H}^{i,N,H}, \mu_{\underline{s}^H}^{\widehat{\boldsymbol{X}}^{N,H}}\right)z\right|^2 \nu(dz)ds \\
&\quad + \int_0^t\int_{\mathbb{R}_0^d} \left[\left|c_\Delta\left(\widehat{X}_{\underline{s}^H}^{i,N,H}, \mu_{\underline{s}^H}^{\widehat{\boldsymbol{X}}^{N,H}}\right)z\right|^2 + 2\left\langle \widehat{X}_{s-}^{i,N,H}, c_\Delta\left(\widehat{X}_{\underline{s}^H}^{i,N,H}, \mu_{\underline{s}^H}^{\widehat{\boldsymbol{X}}^{N,H}}\right)z\right\rangle\right]\widetilde{N}^i(ds,dz) \\
&\leq |x_0|^2 + \int_0^t \Bigg[2\left\langle \widehat{X}_s^{i,N,H} - \widehat{X}_{\underline{s}^H}^{i,N,H}, b_H\left(\widehat{X}_{\underline{s}^H}^{i,N,H}, \mu_{\underline{s}^H}^{\widehat{\boldsymbol{X}}^{N,H}}\right)\right\rangle \\
&\quad + 2\left\langle \widehat{X}_{\underline{s}^H}^{i,N,H}, b_H\left(\widehat{X}_{\underline{s}^H}^{i,N,H}, \mu_{\underline{s}^H}^{\widehat{\boldsymbol{X}}^{N,H}}\right) - b(0,\delta_0)\right\rangle + 2\left\langle \widehat{X}_{\underline{s}^H}^{i,N,H}, b(0,\delta_0)\right\rangle \\
&\quad + \left|\sigma_\Delta\left(\widehat{X}_{\underline{s}^H}^{i,N,H}, \mu_{\underline{s}^H}^{\widehat{\boldsymbol{X}}^{N,H}}\right)\right|^2 + \left|c_\Delta\left(\widehat{X}_{\underline{s}^H}^{i,N,H}, \mu_{\underline{s}^H}^{\widehat{\boldsymbol{X}}^{N,H}}\right)\right|^2 \int_{\mathbb{R}_0^d}|z|^2\nu(dz)\Bigg]ds \\
&\quad + 2\int_0^t \left\langle \widehat{X}_s^{i,N,H} - \widehat{X}_{\underline{s}^H}^{i,N,H}, \sigma_\Delta\left(\widehat{X}_{\underline{s}^H}^{i,N,H}, \mu_{\underline{s}^H}^{\widehat{\boldsymbol{X}}^{N,H}}\right)dW_s^i\right\rangle \\
&\quad + 2\int_0^t \left\langle \widehat{X}_{\underline{s}^H}^{i,N,H}, \sigma_\Delta\left(\widehat{X}_{\underline{s}^H}^{i,N,H}, \mu_{\underline{s}^H}^{\widehat{\boldsymbol{X}}^{N,H}}\right)dW_s^i\right\rangle + \int_0^t\int_{\mathbb{R}_0^d}\Bigg(\left|c_\Delta\left(\widehat{X}_{\underline{s}^H}^{i,N,H}, \mu_{\underline{s}^H}^{\widehat{\boldsymbol{X}}^{N,H}}\right)z\right|^2
\end{aligned}$$

$$+ 2\left\langle \widehat{X}^{i,N,H}_{s-} - \widehat{X}^{i,N,H}_{\underline{s}^H}, c_\Delta\left(\widehat{X}^{i,N,H}_{\underline{s}^H}, \mu^{\widehat{\boldsymbol{X}}^{N,H}}_{\underline{s}^H}\right) z\right\rangle\Bigg) \widetilde{N}^i(ds,dz)$$

$$+ 2\int_0^t \int_{\mathbb{R}^d_0} \left\langle \widehat{X}^{i,N,H}_{\underline{s}^H}, c_\Delta\left(\widehat{X}^{i,N,H}_{\underline{s}^H}, \mu^{\widehat{\boldsymbol{X}}^{N,H}}_{\underline{s}^H}\right) z\right\rangle \widetilde{N}^i(ds,dz)$$

$$\leq |x_0|^2 + \int_0^t \Bigg(2\left|\widehat{X}^{i,N,H}_s - \widehat{X}^{i,N,H}_{\underline{s}^H}\right| \left|b_H\left(\widehat{X}^{i,N,H}_{\underline{s}^H}, \mu^{\widehat{\boldsymbol{X}}^{N,H}}_{\underline{s}^H}\right)\right| + 2C_0\Big(|\widehat{X}^{i,N,H}_{\underline{s}^H}|^2$$

$$+ \mathcal{W}_2^2(\mu^{\widehat{\boldsymbol{X}}^{N,H}}_{\underline{s}^H}, \delta_0)\Big) + |\widehat{X}^{i,N,H}_{\underline{s}^H}|^2 + |b(0,\delta_0)|^2 + \frac{L^2}{\Delta} + \frac{L^2}{\Delta}\int_{\mathbb{R}^d_0} |z|^2\nu(dz)\Bigg) ds$$

$$+ 2\int_0^t \left\langle \widehat{X}^{i,N,H}_s - \widehat{X}^{i,N,H}_{\underline{s}^H}, \sigma_\Delta\left(\widehat{X}^{i,N,H}_{\underline{s}^H}, \mu^{\widehat{\boldsymbol{X}}^{N,H}}_{\underline{s}^H}\right) dW^i_s\right\rangle$$

$$+ 2\int_0^t \left\langle \widehat{X}^{i,N,H}_{\underline{s}^H}, \sigma_\Delta\left(\widehat{X}^{i,N,H}_{\underline{s}^H}, \mu^{\widehat{\boldsymbol{X}}^{N,H}}_{\underline{s}^H}\right) dW^i_s\right\rangle + \int_0^t \int_{\mathbb{R}^d_0} \Bigg(\left|c_\Delta\left(\widehat{X}^{i,N,H}_{\underline{s}^H}, \mu^{\widehat{\boldsymbol{X}}^{N,H}}_{\underline{s}^H}\right) z\right|^2$$

$$+ 2\left\langle \widehat{X}^{i,N,H}_{s-} - \widehat{X}^{i,N,H}_{\underline{s}^H}, c_\Delta\left(\widehat{X}^{i,N,H}_{\underline{s}^H}, \mu^{\widehat{\boldsymbol{X}}^{N,H}}_{\underline{s}^H}\right) z\right\rangle\Bigg) \widetilde{N}^i(ds,dz)$$

$$+ 2\int_0^t \int_{\mathbb{R}^d_0} \left\langle \widehat{X}^{i,N,H}_{\underline{s}^H}, c_\Delta\left(\widehat{X}^{i,N,H}_{\underline{s}^H}, \mu^{\widehat{\boldsymbol{X}}^{N,H}}_{\underline{s}^H}\right) z\right\rangle \widetilde{N}^i(ds,dz). \tag{3.25}$$

Now, we define $\tau_R := \inf\{t > 0 : \max_{i\in\{1,\ldots,N\}} |\widehat{X}^{i,N,H}_t| > R\}$ for each $R > 0$ and $\tau := s \wedge \tau_R$. On the one hand, using equation (3.24), Condition **T3**, the isometry property of stochastic integrals and the fact that $|b_H(x,\mu)|h^H(x,\mu) \leq C_0$, we get

$$\mathbb{E}\left[\left|\widehat{X}^{i,N,H}_\tau - \widehat{X}^{i,N,H}_{\underline{\tau}^H}\right|^2\right] \leq 3\Bigg(\mathbb{E}\Bigg[\left|b_H\left(\widehat{X}^{i,N,H}_{\underline{\tau}^H}, \mu^{\widehat{\boldsymbol{X}}^{N,H}}_{\underline{\tau}^H}\right)\right|^2 (\tau - \underline{\tau}^H)^2\Bigg]$$

$$+ \left|\sigma_\Delta\left(\widehat{X}^{i,N,H}_{\underline{\tau}^H}, \mu^{\widehat{\boldsymbol{X}}^{N,H}}_{\underline{\tau}^H}\right)\right|^2 \left|W^i_\tau - W^i_{\underline{\tau}^H}\right|^2 + \left|c_\Delta\left(\widehat{X}^{i,N,H}_{\underline{\tau}^H}, \mu^{\widehat{\boldsymbol{X}}^{N,H}}_{\underline{\tau}^H}\right)\right|^2 \left|Z^i_\tau - Z^i_{\underline{\tau}^H}\right|^2\Bigg)$$

$$\leq 3\left(C_0^2\Delta^2 + \frac{L^2}{\Delta}\mathbb{E}\left[\tau - \underline{\tau}^H\right] + \frac{L^2}{\Delta}\int_{\mathbb{R}^d_0} |z|^2\nu(dz)\mathbb{E}\left[\tau - \underline{\tau}^H\right]\right)$$

$$\leq 3\left(C_0^2\Delta^2 + L^2 + L^2\int_{\mathbb{R}^d_0} |z|^2\nu(dz)\right). \tag{3.26}$$

On the other hand, Condition **T3** yields to

$$\left|\sigma_\Delta\left(\widehat{X}^{i,N,H}_{\underline{s}^H}, \mu^{\widehat{\boldsymbol{X}}^{N,H}}_{\underline{s}^H}\right)\right| \leq \frac{L}{\sqrt{\Delta}}, \quad \left|c_\Delta\left(\widehat{X}^{i,N,H}_{\underline{s}^H}, \mu^{\widehat{\boldsymbol{X}}^{N,H}}_{\underline{s}^H}\right)\right| \leq \frac{L}{\sqrt{\Delta}},$$

$$\left|\widehat{X}^{i,N,H}_{\underline{s}^H}\right| \left|\sigma_\Delta\left(\widehat{X}^{i,N,H}_{\underline{s}^H}, \mu^{\widehat{\boldsymbol{X}}^{N,H}}_{\underline{s}^H}\right)\right| \leq \frac{L}{\sqrt{\Delta}}, \quad \left|\widehat{X}^{i,N,H}_{\underline{s}^H}\right| \left|c_\Delta\left(\widehat{X}^{i,N,H}_{\underline{s}^H}, \mu^{\widehat{\boldsymbol{X}}^{N,H}}_{\underline{s}^H}\right)\right| \leq \frac{L}{\sqrt{\Delta}}. \tag{3.27}$$

Therefore, all the stochastic integrals with respect to the Brownian motion and the compensated Poisson random measure above are square martingales. Thus, their moments are equal to zero.

Moreover, using **T3**, equation (3.24), moment properties of the Brownian motion, the isometry property of stochastic integrals and the fact that $|b_H(x,\mu)|^2 h^H(x,\mu) \leq C_0$, we get

$$
\begin{aligned}
&\mathbb{E}\left[\left|\widehat{X}^{i,N,H}_{s} - \widehat{X}^{i,N,H}_{\underline{s}^H}\right| \left|b_H\left(\widehat{X}^{i,N,H}_{\underline{s}^H}, \mu^{\widehat{\mathbf{X}}^{N,H}}_{\underline{s}^H}\right)\right| \Big| \mathcal{F}_{\underline{s}^H}\right] \\
&\leq \mathbb{E}\Bigg[\left|b_H\left(\widehat{X}^{i,N,H}_{\underline{s}^H}, \mu^{\widehat{\mathbf{X}}^{N,H}}_{\underline{s}^H}\right)\right|^2 (s-\underline{s}^H) + \left|b_H\left(\widehat{X}^{i,N,H}_{\underline{s}^H}, \mu^{\widehat{\mathbf{X}}^{N,H}}_{\underline{s}^H}\right)\right| \left|\sigma_\Delta\left(\widehat{X}^{i,N,H}_{\underline{s}^H}, \mu^{\widehat{\mathbf{X}}^{N,H}}_{\underline{s}^H}\right)\right| \\
&\quad \times \left|W^i_s - W^i_{\underline{s}^H}\right| + \left|b_H\left(\widehat{X}^{i,N,H}_{\underline{s}^H}, \mu^{\widehat{\mathbf{X}}^{N,H}}_{\underline{s}^H}\right)\right| \left|c_\Delta\left(\widehat{X}^{i,N,H}_{\underline{s}^H}, \mu^{\widehat{\mathbf{X}}^{N,H}}_{\underline{s}^H}\right)\right| \left|Z^i_s - Z^i_{\underline{s}^H}\right| \Big| \mathcal{F}_{\underline{s}^H}\Bigg] \\
&\leq \left|b_H\left(\widehat{X}^{i,N,H}_{\underline{s}^H}, \mu^{\widehat{\mathbf{X}}^{N,H}}_{\underline{s}^H}\right)\right|^2 (s-\underline{s}^H) + \frac{L}{\sqrt{\Delta}} \left|b_H\left(\widehat{X}^{i,N,H}_{\underline{s}^H}, \mu^{\widehat{\mathbf{X}}^{N,H}}_{\underline{s}^H}\right)\right| \sqrt{s-\underline{s}^H} \\
&\quad + \frac{L}{\sqrt{\Delta}} \left(\int_{\mathbb{R}^d_0} |z|^2 \nu(dz)\right)^{1/2} \left|b_H\left(\widehat{X}^{i,N,H}_{\underline{s}^H}, \mu^{\widehat{\mathbf{X}}^{N,H}}_{\underline{s}^H}\right)\right| \sqrt{s-\underline{s}^H} \\
&\leq C_0\Delta + L\sqrt{C_0} + L\sqrt{C_0}\left(\int_{\mathbb{R}^d_0} |z|^2 \nu(dz)\right)^{1/2}.
\end{aligned} \tag{3.28}
$$

This, combined with $\mathbb{E}\left[\mathcal{W}_2^2(\mu^{\widehat{\mathbf{X}}^{N,H}}_{\underline{s}^H}, \delta_0)\mathbf{1}_{s\leq\tau_R}\right] = \mathbb{E}\left[|\widehat{X}^{i,N,H}_{\underline{s}^H}|^2 \mathbf{1}_{s\leq\tau_R}\right]$ for $i \in \{1,\dots,N\}$, yields that for any $t \in (0,T]$,

$$
\mathbb{E}\left[\left|\widehat{X}^{i,N,H}_{t\wedge\tau_R}\right|^2\right] \leq |x_0|^2 + \int_0^t C\left(L, \Delta, b(0,\delta_0), \mu_2\right)\left(1 + \mathbb{E}\left[\left|\widehat{X}^{i,N,H}_{\underline{s}^H}\right|^2 \mathbf{1}_{s\leq\tau_R}\right]\right) ds,
$$

where $\mu_2 := \int_{\mathbb{R}^d_0} |z|^2 \nu(dz)$.

Next, using equation (3.24) and (3.26), we get

$$
\begin{aligned}
\mathbb{E}\left[\left|\widehat{X}^{i,N,H}_{\underline{s}^H}\right|^2 \mathbf{1}_{s\leq\tau_R}\right] &\leq 2\mathbb{E}\left[\left|\widehat{X}^{i,N,H}_{s}\right|^2 \mathbf{1}_{s\leq\tau_R}\right] + 2\mathbb{E}\left[\left|\widehat{X}^{i,N,H}_{s} - \widehat{X}^{i,N,H}_{\underline{s}^H}\right|^2\right] \\
&\leq 2\mathbb{E}\left[\left|\widehat{X}^{i,N,H}_{s\wedge\tau_R}\right|^2\right] + 6\left(C_0^2\Delta^2 + L^2 + L^2\mu_2\right).
\end{aligned}
$$

This implies that for any $t \in (0,T]$,

$$
\mathbb{E}\left[\left|\widehat{X}^{i,N,H}_{t\wedge\tau_R}\right|^2\right] \leq C\left(x_0, L, \Delta, b(0,\delta_0), \mu_2, T\right)\left(1 + \int_0^t \mathbb{E}\left[\left|\widehat{X}^{i,N,H}_{s\wedge\tau_R}\right|^2\right] ds\right),
$$

which, together with Gronwall's inequality, yields that for any $t \in (0,T]$,

$$
\max_{i\in\{1,\dots,N\}} \sup_{t\in[0,T]} \mathbb{E}\left[\left|\widehat{X}^{i,N,H}_{t\wedge\tau_R}\right|^2\right] \leq C\left(x_0, L, \Delta, b(0,\delta_0), \mu_2, T\right).
$$

Then, using Markov's inequality, we obtain that

$$\begin{aligned}
\mathbb{P}(\tau_R < T) &\leq \sum_{i=1}^{N} \mathbb{P}(|\widehat{X}^{i,N,H}_{T\wedge\tau_R}| > R) = N\mathbb{P}(|\widehat{X}^{1,N,H}_{T\wedge\tau_R}| > R) \\
&\leq N\frac{\mathbb{E}[|\widehat{X}^{1,N,H}_{T\wedge\tau_R}|^2]}{R^2} \\
&\leq \frac{NC\left(x_0, L, \Delta, b(0,\delta_0), \mu_2, T\right)}{R^2},
\end{aligned}$$

which tends to zero as $R \uparrow \infty$. Therefore, $\tau_R \uparrow \infty$ as $R \uparrow \infty$. Then due to Fatou's lemma, we get

$$\max_{i\in\{1,\dots,N\}} \sup_{t\in[0,T]} \mathbb{E}\left[\left|\widehat{X}^{i,N,H}_t\right|^2\right] \leq C\left(x_0, L, \Delta, b(0,\delta_0), \mu_2, T\right). \tag{3.29}$$

Now, from (3.25), we get that for any $t \in (0,T]$,

$$\begin{aligned}
&|\widehat{X}^{i,N,H}_t|^2 \\
&\leq |x_0|^2 + \int_0^T \Bigg[2\left|\widehat{X}^{i,N,H}_s - \widehat{X}^{i,N,H}_{\underline{s}^H}\right| \left|b_H\left(\widehat{X}^{i,N,H}_{\underline{s}^H}, \mu^{\widehat{\boldsymbol{X}}^{N,H}}_{\underline{s}^H}\right)\right| + |\widehat{X}^{i,N,H}_{\underline{s}^H}|^2 \\
&\quad + 2C_0\left(|\widehat{X}^{i,N,H}_{\underline{s}^H}|^2 + \mathcal{W}_2^2(\mu^{\widehat{\boldsymbol{X}}^{N,H}}_{\underline{s}^H}, \delta_0)\right) + |b(0,\delta_0)|^2 + \frac{L^2}{\Delta} + \frac{L^2}{\Delta}\mu_2\Bigg] ds \\
&\quad + 2\int_0^t \left\langle \widehat{X}^{i,N,H}_s - \widehat{X}^{i,N,H}_{\underline{s}^H}, \sigma_\Delta\left(\widehat{X}^{i,N,H}_{\underline{s}^H}, \mu^{\widehat{\boldsymbol{X}}^{N,H}}_{\underline{s}^H}\right) dW^i_s\right\rangle \\
&\quad + 2\int_0^t \left\langle \widehat{X}^{i,N,H}_{\underline{s}^H}, \sigma_\Delta\left(\widehat{X}^{i,N,H}_{\underline{s}^H}, \mu^{\widehat{\boldsymbol{X}}^{N,H}}_{\underline{s}^H}\right) dW^i_s\right\rangle + \int_0^t \int_{\mathbb{R}^d_0} \Bigg[\left|c_\Delta\left(\widehat{X}^{i,N,H}_{\underline{s}^H}, \mu^{\widehat{\boldsymbol{X}}^{N,H}}_{\underline{s}^H}\right) z\right|^2 \\
&\quad + 2\left\langle \widehat{X}^{i,N,H}_{s-} - \widehat{X}^{i,N,H}_{\underline{s}^H}, c_\Delta\left(\widehat{X}^{i,N,H}_{\underline{s}^H}, \mu^{\widehat{\boldsymbol{X}}^{N,H}}_{\underline{s}^H}\right) z\right\rangle\Bigg] \widetilde{N}^i(ds,dz) \\
&\quad + 2\int_0^t \int_{\mathbb{R}^d_0} \left\langle \widehat{X}^{i,N,H}_{\underline{s}^H}, c_\Delta\left(\widehat{X}^{i,N,H}_{\underline{s}^H}, \mu^{\widehat{\boldsymbol{X}}^{N,H}}_{\underline{s}^H}\right) z\right\rangle \widetilde{N}^i(ds,dz).
\end{aligned}$$

This, combined with (3.28), the fact that $\mathbb{E}\left[\mathcal{W}_2^2(\mu^{\widehat{\boldsymbol{X}}^{N,H}}_{\underline{s}^H}, \delta_0)\right] = \mathbb{E}\left[|\widehat{X}^{i,N,H}_{\underline{s}^H}|^2\right]$ and (3.29), deduces that

$$\max_{i\in\{1,\dots,N\}} \mathbb{E}\left[\sup_{t\in[0,T]} \left|\widehat{X}^{i,N,H}_t\right|^2\right] \leq C\left(x_0, L, \Delta, b(0,\delta_0), \mu_2, T\right) =: \overline{C_0}. \tag{3.30}$$

Observe that

$$\{t_k \leq T\} = \left\{t_k \leq T,\ \max_{i\in\{1,\dots,N\}} \sup_{t\in[0,T]} \left(\left|\widehat{X}^{i,N,H}_t\right|^{\beta_1} + \mathcal{W}_2^{\beta_2}\left(\mu^{\widehat{\boldsymbol{X}}^{N,H}}_t, \delta_0\right)\right) \leq \frac{H}{2}\right\}$$

$$\cup \left\{ \max_{i\in\{1,\dots,N\}} \sup_{t\in[0,T]} \left(\left| \widehat{X}_t^{i,N,H} \right|^{\beta_1} + \mathcal{W}_2^{\beta_2}\left(\mu_t^{\widehat{\boldsymbol{X}}^{N,H}}, \delta_0 \right) \right) > \frac{H}{2} \right\}$$

$$\subset \left\{ t_k^H \leq T \right\} \cup \left\{ \max_{i\in\{1,\dots,N\}} \sup_{t\in[0,T]} \left(\left| \widehat{X}_t^{i,N,H} \right|^{\beta_1} + \mathcal{W}_2^{\beta_2}\left(\mu_t^{\widehat{\boldsymbol{X}}^{N,H}}, \delta_0 \right) \right) > \frac{H}{2} \right\}.$$

Then, using $\mathbb{E}\left[\sup_{t\in[0,T]} \mathcal{W}_2^2(\mu_t^{\widehat{\boldsymbol{X}}^{N,H}}, \delta_0)\right] \leq \mathbb{E}\left[\sup_{t\in[0,T]} |\widehat{X}_t^{i,N,H}|^2\right]$ for $i \in \{1,\dots,N\}$ and Markov's inequality, we get that for any $H > 0$,

$$\begin{aligned}
&\mathbb{P}(t_k \leq T) \\
&\leq \mathbb{P}\left(t_k^H \leq T\right) + \mathbb{P}\left(\max_{i\in\{1,\dots,N\}} \sup_{t\in[0,T]} \left(\left| \widehat{X}_t^{i,N,H} \right|^{\beta_1} + \mathcal{W}_2^{\beta_2}\left(\mu_t^{\widehat{\boldsymbol{X}}^{N,H}}, \delta_0 \right) \right) > \frac{H}{2} \right) \\
&\leq \mathbb{P}\left(t_k^H \leq T\right) + \mathbb{P}\left(\max_{i\in\{1,\dots,N\}} \sup_{t\in[0,T]} \left| \widehat{X}_t^{i,N,H} \right|^{\beta_1} > \frac{H}{4} \right) \\
&\quad + \mathbb{P}\left(\max_{i\in\{1,\dots,N\}} \sup_{t\in[0,T]} \mathcal{W}_2^{\beta_2}\left(\mu_t^{\widehat{\boldsymbol{X}}^{N,H}}, \delta_0 \right) > \frac{H}{4} \right) \\
&= \mathbb{P}\left(t_k^H \leq T\right) + \mathbb{P}\left(\max_{i\in\{1,\dots,N\}} \sup_{t\in[0,T]} \left| \widehat{X}_t^{i,N,H} \right|^{2} > \left(\frac{H}{4}\right)^{2/\beta_1} \right) \\
&\quad + \mathbb{P}\left(\max_{i\in\{1,\dots,N\}} \sup_{t\in[0,T]} \mathcal{W}_2^{2}\left(\mu_t^{\widehat{\boldsymbol{X}}^{N,H}}, \delta_0 \right) > \left(\frac{H}{4}\right)^{2/\beta_2} \right) \\
&\leq \mathbb{P}\left(t_k^H \leq T\right) + \left(\frac{4}{H}\right)^{2/\beta_1} \sum_{i=1}^N \mathbb{E}\left[\sup_{t\in[0,T]} \left| \widehat{X}_t^{i,N,H} \right|^2 \right] \\
&\quad + \left(\frac{4}{H}\right)^{2/\beta_2} \sum_{i=1}^N \mathbb{E}\left[\sup_{t\in[0,T]} \mathcal{W}_2^2(\mu_t^{\widehat{\boldsymbol{X}}^{N,H}}, \delta_0) \right] \\
&\leq \mathbb{P}\left(t_k^H \leq T\right) + \left(\frac{4}{H}\right)^{2/\beta_1} \sum_{i=1}^N \mathbb{E}\left[\sup_{t\in[0,T]} \left| \widehat{X}_t^{i,N,H} \right|^2 \right] \\
&\quad + \left(\frac{4}{H}\right)^{2/\beta_2} \sum_{i=1}^N \mathbb{E}\left[\sup_{t\in[0,T]} \left| \widehat{X}_t^{i,N,H} \right|^2 \right] \\
&\leq \mathbb{P}\left(t_k^H \leq T\right) + \left(\left(\frac{4}{H}\right)^{2/\beta_1} + \left(\frac{4}{H}\right)^{2/\beta_2} \right) N\overline{C_0}.
\end{aligned}$$

Then, let $k \uparrow \infty$ and recall that $\lim_{k\to+\infty} t_k^H = +\infty$ a.s., we have that for any $H > 0$,

$$\limsup_{k\to\infty} \mathbb{P}(t_k \leq T) \leq \left(\left(\frac{4}{H}\right)^{2/\beta_1} + \left(\frac{4}{H}\right)^{2/\beta_2} \right) N\overline{C_0}.$$

Then, letting $H \uparrow \infty$, we get

$$\lim_{k\to\infty} \mathbb{P}(t_k \le T) = 0.$$

Therefore, $t_k \to \infty$ in probability as $k \uparrow \infty$. Since $(t_k)_{k\ge 0}$ is an increasing sequence, we have

$$\lim_{k\to+\infty} t_k = +\infty \quad \text{a.s.}$$

Thus, the result follows. □

Remark 3.5.2. The demonstration that t_k diverges to infinity is a critical and challenging aspect of constructing an adaptive scheme. Our approach to proving Proposition 3.5.1 diverges significantly from that presented in [18]. Specifically, in [18], the auxiliary process $\widehat{X}$ is formulated using the projection method, which complicates its analysis within the context of McKean-Vlasov SDEs. In our proof, we also employ a truncation technique, but instead of applying it to the entire process $\widehat{X}$, we truncate the coefficient b and the step size h. This modification results in the auxiliary process $\widehat{X}^{i,N,H}$ being constructed as an Itô process, enabling us to apply the Itô formula to $|\widehat{X}^{i,N,H}|^2$. This approach substantially streamlines our proof.

Let all assumptions of Proposition 3.5.1 be satisfied, we define the nearest time point before t by $\underline{t} := \max\{t_n : t_n \le t\}$, and by $N_t := \max\{n : t_n \le t\}$ the number of timesteps approximation up to time t. Observe that $\underline{t}$ is a stopping time. Thus, we define the standard continuous interpolant as

$$\begin{aligned}\widehat{X}_t^{i,N} = \widehat{X}_{\underline{t}}^{i,N} &+ b\left(\widehat{X}_{\underline{t}}^{i,N}, \mu_{\underline{t}}^{\widehat{\boldsymbol{X}}^N}\right)(t-\underline{t}) + \sigma_\Delta\left(\widehat{X}_{\underline{t}}^{i,N}, \mu_{\underline{t}}^{\widehat{\boldsymbol{X}}^N}\right)\left(W_t^i - W_{\underline{t}}^i\right)\\ &+ c_\Delta\left(\widehat{X}_{\underline{t}}^{i,N}, \mu_{\underline{t}}^{\widehat{\boldsymbol{X}}^N}\right)\left(Z_t^i - Z_{\underline{t}}^i\right). \end{aligned} \tag{3.31}$$

Hence, $\widehat{X}^{i,N} = (\widehat{X}_t^{i,N})_{t\ge 0}$ is the solution to the following SDE with jumps

$$\begin{aligned} d\widehat{X}_t^{i,N} &= b\left(\widehat{X}_{\underline{t}}^{i,N}, \mu_{\underline{t}}^{\widehat{\boldsymbol{X}}^N}\right)dt + \sigma_\Delta\left(\widehat{X}_{\underline{t}}^{i,N}, \mu_{\underline{t}}^{\widehat{\boldsymbol{X}}^N}\right)dW_t^i + c_\Delta\left(\widehat{X}_{\underline{t}}^{i,N}, \mu_{\underline{t}}^{\widehat{\boldsymbol{X}}^N}\right)dZ_t^i,\\ \widehat{X}_0^{i,N} &= x_0, \end{aligned} \tag{3.32}$$

whose integral equation has the following form

$$\begin{aligned}\widehat{X}_t^{i,N} = x_0 &+ \int_0^t b\left(\widehat{X}_{\underline{s}}^{i,N}, \mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N}\right)ds + \int_0^t \sigma_\Delta\left(\widehat{X}_{\underline{s}}^{i,N}, \mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N}\right)dW_s^i\\ &+ \int_0^t\int_{\mathbb{R}_0^d} c_\Delta\left(\widehat{X}_{\underline{s}}^{i,N}, \mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N}\right) z\widetilde{N}^i(ds, dz).\end{aligned}$$

3.6 Moments

In Section 3.3, we demonstrated the integrability of the moment of the exact solution in L^p for any $p \in [2, p_0]$. Notably, when $\gamma < 0$, the moments of the exact solution are uniformly bounded over infinite time intervals. In this section, we will establish that the moments of the approximate solution exhibit similar properties to those of the exact solution. Our proof techniques remain consistent, utilizing mathematical induction, the binomial theorem, and Hölder inequality. However, due to the two-step approximation process required for McKeanVlasov stochastic differential equations, proving the integrability of the approximate solution is more challenging than in the standard SDE case (Chapter 2). Consequently, we require several auxiliary estimates.

Firstly, we state the estimate on the moments of $\widehat{X}^{i,N} = (\widehat{X}_t^{i,N})_{t\geq 0}$.

Lemma 3.6.1. *Assume Conditions* ***T1*–*T3*** *hold and that Condition* ***C5*** *holds for* $q = 2p_0$*. Then for any* $p \in [1, 2p_0]$ *and* $T > 0$*, there exists a positive constant* $C\left(p, x_0, L, \Delta, b(0,\delta_0), \mu_2, \mu_{p/2}, T\right)$ *with* $\mu_{p/2} := \int_{\mathbb{R}_0^d} |z|^{p/2}\nu(dz)$ *such that*

$$\max_{i\in\{1,\dots,N\}} \mathbb{E}\left[\sup_{t\in[0,T]} |\widehat{X}_t^{i,N}|^p\right] \leq C\left(p, x_0, L, \Delta, b(0,\delta_0), \mu_2, \mu_{p/2}, T\right).$$

Proof. Recall that the process $\widehat{X}^{i,N,H} = (\widehat{X}_t^{i,N,H})_{t\geq 0}$ is defined in (3.23) and (3.24). Using Markov's inequality, the estimate $\mathbb{E}\left[\sup_{t\in[0,T]} \mathcal{W}_2^2(\mu_t^{\widehat{\boldsymbol{X}}^{N,H}}, \delta_0)\right] \leq \mathbb{E}\left[\sup_{t\in[0,T]} |\widehat{X}_t^{i,N,H}|^2\right]$ and (3.30), we obtain that for any $T > 0$, $i \in \{1, \dots, N\}$ and $H > 0$,

$$\begin{aligned}
&\mathbb{P}\left(\sup_{t\in[0,T]} |\widehat{X}_t^{i,N}| \neq \sup_{t\in[0,T]} |\widehat{X}_t^{i,N,H}|\right) \\
&\leq \mathbb{P}\left(\sup_{t\in[0,T]} \left(\left|\widehat{X}_t^{i,N,H}\right|^{\beta_1} + \mathcal{W}_2^{\beta_2}\left(\mu_t^{\widehat{\boldsymbol{X}}^{N,H}}, \delta_0\right)\right) > H\right) \\
&\leq \mathbb{P}\left(\sup_{t\in[0,T]} \left|\widehat{X}_t^{i,N,H}\right|^{\beta_1} > \frac{H}{2}\right) + \mathbb{P}\left(\sup_{t\in[0,T]} \mathcal{W}_2^{\beta_2}\left(\mu_t^{\widehat{\boldsymbol{X}}^{N,H}}, \delta_0\right) > \frac{H}{2}\right) \\
&= \mathbb{P}\left(\sup_{t\in[0,T]} \left|\widehat{X}_t^{i,N,H}\right|^{2} > \left(\frac{H}{2}\right)^{2/\beta_1}\right) + \mathbb{P}\left(\sup_{t\in[0,T]} \mathcal{W}_2^{2}\left(\mu_t^{\widehat{\boldsymbol{X}}^{N,H}}, \delta_0\right) > \left(\frac{H}{2}\right)^{2/\beta_2}\right) \\
&\leq \left(\frac{2}{H}\right)^{2/\beta_1} \mathbb{E}\left[\sup_{t\in[0,T]} \left|\widehat{X}_t^{i,N,H}\right|^2\right] + \left(\frac{2}{H}\right)^{2/\beta_2} \mathbb{E}\left[\sup_{t\in[0,T]} \mathcal{W}_2^2(\mu_t^{\widehat{\boldsymbol{X}}^{N,H}}, \delta_0)\right] \\
&\leq \left(\frac{2}{H}\right)^{2/\beta_1} \mathbb{E}\left[\sup_{t\in[0,T]} \left|\widehat{X}_t^{i,N,H}\right|^2\right] + \left(\frac{2}{H}\right)^{2/\beta_2} \mathbb{E}\left[\sup_{t\in[0,T]} \left|\widehat{X}_t^{i,N,H}\right|^2\right]
\end{aligned}$$

$$\leq \left(\left(\frac{2}{H}\right)^{2/\beta_1}+\left(\frac{2}{H}\right)^{2/\beta_2}\right)\overline{C_0},$$

which tends to zero as $H\uparrow\infty$. This implies that $\sup_{t\in[0,T]}|\widehat{X}^{i,N,H}_t|\to\sup_{t\in[0,T]}|\widehat{X}^{i,N}_t|$ in probability as $H\uparrow\infty$. Thus, for any $T>0$ and $i\in\{1,\ldots,N\}$, there exists a sequence $\{H_n\}_{n\geq1}$ that tends to infinity such that $\sup_{t\in[0,T]}|\widehat{X}^{i,N,H_n}_t|\to\sup_{t\in[0,T]}|\widehat{X}^{i,N}_t|$ a.s. as $n\uparrow\infty$.

Now, from (3.25), we have that for any $t>0$, $i\in\{1,\ldots,N\}$ and $H>0$,

$$\begin{aligned}
&|\widehat{X}^{i,N,H}_t|^2\\
&\leq |x_0|^2+\int_0^T\Bigg[2\left|\widehat{X}^{i,N,H}_s-\widehat{X}^{i,N,H}_{\underline{s}^H}\right|\left|b_H\left(\widehat{X}^{i,N,H}_{\underline{s}^H},\mu^{\widehat{\boldsymbol{X}}^{N,H}}_{\underline{s}^H}\right)\right|+2C_0\mathcal{W}_2^2(\mu^{\widehat{\boldsymbol{X}}^{N,H}}_{\underline{s}^H},\delta_0)\\
&\quad+|b(0,\delta_0)|^2+\frac{L^2}{\Delta}+\frac{L^2}{\Delta}\int_{\mathbb{R}^d_0}|z|^2\nu(dz)\Bigg]ds+2\int_0^t\left\langle\widehat{X}^{i,N,H}_{\underline{s}^H},\sigma_\Delta\left(\widehat{X}^{i,N,H}_{\underline{s}^H},\mu^{\widehat{\boldsymbol{X}}^{N,H}}_{\underline{s}^H}\right)dW^i_s\right\rangle\\
&\quad+2\int_0^t\left\langle\widehat{X}^{i,N,H}_s-\widehat{X}^{i,N,H}_{\underline{s}^H},\sigma_\Delta\left(\widehat{X}^{i,N,H}_{\underline{s}^H},\mu^{\widehat{\boldsymbol{X}}^{N,H}}_{\underline{s}^H}\right)dW^i_s\right\rangle\\
&\quad+\int_0^t\int_{\mathbb{R}^d_0}\left[\left|c_\Delta\left(\widehat{X}^{i,N,H}_{\underline{s}^H},\mu^{\widehat{\boldsymbol{X}}^{N,H}}_{\underline{s}^H}\right)z\right|^2+2\left\langle\widehat{X}^{i,N,H}_{s-}-\widehat{X}^{i,N,H}_{\underline{s}^H},c_\Delta\left(\widehat{X}^{i,N,H}_{\underline{s}^H},\mu^{\widehat{\boldsymbol{X}}^{N,H}}_{\underline{s}^H}\right)z\right\rangle\right]\\
&\quad\times\widetilde{N}^i(ds,dz)+2\int_0^t\int_{\mathbb{R}^d_0}\left\langle\widehat{X}^{i,N,H}_{\underline{s}^H},c_\Delta\left(\widehat{X}^{i,N,H}_{\underline{s}^H},\mu^{\widehat{\boldsymbol{X}}^{N,H}}_{\underline{s}^H}\right)z\right\rangle\widetilde{N}^i(ds,dz)\\
&\quad+(2C_0+1)\int_0^t|\widehat{X}^{i,N,H}_{\underline{s}^H}|^2ds. \qquad (3.33)
\end{aligned}$$

First, we have

$$\begin{aligned}
&\mathbb{E}\left[\left|\widehat{X}^{i,N,H}_s-\widehat{X}^{i,N,H}_{\underline{s}^H}\right|^{p/2}\left|b_H\left(\widehat{X}^{i,N,H}_{\underline{s}^H},\mu^{\widehat{\boldsymbol{X}}^{N,H}}_{\underline{s}^H}\right)\right|^{p/2}\Big|\mathcal{F}_{\underline{s}^H}\right]\\
&\leq C(p)\Bigg[\mathbb{E}\Bigg[\left|b_H\left(\widehat{X}^{i,N,H}_{\underline{s}^H},\mu^{\widehat{\boldsymbol{X}}^{N,H}}_{\underline{s}^H}\right)\right|^p(s-\underline{s}^H)^{p/2}\\
&\quad+\left|b_H\left(\widehat{X}^{i,N,H}_{\underline{s}^H},\mu^{\widehat{\boldsymbol{X}}^{N,H}}_{\underline{s}^H}\right)\right|^{p/2}\left|\sigma_\Delta\left(\widehat{X}^{i,N,H}_{\underline{s}^H},\mu^{\widehat{\boldsymbol{X}}^{N,H}}_{\underline{s}^H}\right)\right|^{p/2}\left|W^i_s-W^i_{\underline{s}^H}\right|^{p/2}\\
&\quad+\left|b_H\left(\widehat{X}^{i,N,H}_{\underline{s}^H},\mu^{\widehat{\boldsymbol{X}}^{N,H}}_{\underline{s}^H}\right)\right|^{p/2}\left|c_\Delta\left(\widehat{X}^{i,N,H}_{\underline{s}^H},\mu^{\widehat{\boldsymbol{X}}^{N,H}}_{\underline{s}^H}\right)\right|^{p/2}\left|Z^i_s-Z^i_{\underline{s}^H}\right|^{p/2}\Big|\mathcal{F}_{\underline{s}^H}\Bigg]\Bigg]\\
&\leq C(p)\Bigg[\left|b_H\left(\widehat{X}^{i,N,H}_{\underline{s}^H},\mu^{\widehat{\boldsymbol{X}}^{N,H}}_{\underline{s}^H}\right)\right|^p(s-\underline{s}^H)^{p/2}+\left(\frac{L}{\sqrt{\Delta}}\right)^{p/2}\int_{\mathbb{R}^d_0}|z|^{p/2}\nu(dz)(s-\underline{s}^H)\\
&\quad+\left(\frac{L}{\sqrt{\Delta}}\right)^{p/2}\left(\left|b_H\left(\widehat{X}^{i,N,H}_{\underline{s}^H},\mu^{\widehat{\boldsymbol{X}}^{N,H}}_{\underline{s}^H}\right)\right|^2(s-\underline{s}^H)\right)^{p/4}\Bigg]
\end{aligned}$$

$$\leq C(p)\left[C_0^{p/2}\Delta^{p/2}+L^{p/2}C_0^{p/4}+\left(\frac{L}{\sqrt{\Delta}}\right)^{p/2}\int_{\mathbb{R}_0^d}|z|^{p/2}\nu(dz)\right]. \tag{3.34}$$

Second, we have

$$\begin{aligned}
&\mathbb{E}\left[\left|\widehat{X}_{\tau}^{i,N,H}-\widehat{X}_{\underline{\tau}^H}^{i,N,H}\right|^{p/2}\right]\\
&\leq 3^{p/2-1}\left[\mathbb{E}\left[\left|b_H\left(\widehat{X}_{\underline{\tau}^H}^{i,N,H},\mu_{\underline{\tau}^H}^{\widehat{\boldsymbol{X}}^{N,H}}\right)\right|^{p/2}(\tau-\underline{\tau}^H)^{p/2}\right]\right.\\
&\left.+\left|\sigma_\Delta\left(\widehat{X}_{\underline{\tau}^H}^{i,N,H},\mu_{\underline{\tau}^H}^{\widehat{\boldsymbol{X}}^{N,H}}\right)\right|^{p/2}\left|W_\tau^i-W_{\underline{\tau}^H}^i\right|^{p/2}+\left|c_\Delta\left(\widehat{X}_{\underline{\tau}^H}^{i,N,H},\mu_{\underline{\tau}^H}^{\widehat{\boldsymbol{X}}^{N,H}}\right)\right|^{p/2}\left|Z_\tau^i-Z_{\underline{\tau}^H}^i\right|^{p/2}\right]\\
&\leq C_p\left[C_0^{p/2}\Delta^{p/2}+\left(\frac{L}{\sqrt{\Delta}}\right)^{p/2}\mathbb{E}\left[(\tau-\underline{\tau}^H)^{p/4}\right]+\left(\frac{L}{\sqrt{\Delta}}\right)^{p/2}\int_{\mathbb{R}_0^d}|z|^{p/2}\nu(dz)\mathbb{E}\left[\tau-\underline{\tau}^H\right]\right]\\
&\leq C_p\left[C_0^{p/2}\Delta^{p/2}+L^{p/2}+\left(\frac{L}{\sqrt{\Delta}}\right)^{p/2}\int_{\mathbb{R}_0^d}|z|^{p/2}\nu(dz)\right].
\end{aligned}\tag{3.35}$$

Therefore, by using the estimate $\mathbb{E}\left[\mathcal{W}_2^2(\mu_{\underline{s}^H}^{\widehat{\boldsymbol{X}}^{N,H}},\delta_0)\right]=\mathbb{E}\left[|\widehat{X}_{\underline{s}^H}^{i,N,H}|^2\right]\leq\overline{C_0}$; estimates (3.33), (3.34), (3.35), (3.27) and the Burkholder-Davis-Gundy inequality, we get that for any $t\in(0,T]$,

$$\begin{aligned}
&\mathbb{E}\left[\sup_{u\in[0,t]}|\widehat{X}_u^{i,N,H}|^p\right]\\
&\leq C\left(p,x_0,L,\Delta,b(0,\delta_0),\mu_2,\mu_{p/2},T\right)+(2C_0+1)^{p/2}\mathbb{E}\left[\left(\int_0^t|\widehat{X}_{\underline{s}^H}^{i,N,H}|^2ds\right)^{p/2}\right]\\
&\leq\widetilde{C}_0+(2C_0+1)^{p/2}t^{p/2-1}\int_0^t\mathbb{E}\left[|\widehat{X}_{\underline{s}^H}^{i,N,H}|^p\right]ds\\
&\leq\widetilde{C}_0+(2C_0+1)^{p/2}t^{p/2-1}\int_0^t\mathbb{E}\left[\sup_{u\in[0,s]}|\widehat{X}_u^{i,N,H}|^p\right]ds,
\end{aligned}$$

where $\widetilde{C}_0:=C\left(p,x_0,L,\Delta,b(0,\delta_0),\mu_2,\mu_{p/2},T\right)$ with $\mu_{p/2}:=\int_{\mathbb{R}_0^d}|z|^{p/2}\nu(dz)$.

This, combined with Gronwall's inequality, deduces that

$$\max_{i\in\{1,\ldots,N\}}\mathbb{E}\left[\sup_{t\in[0,T]}|\widehat{X}_t^{i,N,H}|^p\right]\leq\widetilde{C}_1, \tag{3.36}$$

where the constant $\widetilde{C}_1$ does not depend on H.

Therefore, choosing $H = H_n$ in (3.36) and letting $n \uparrow \infty$, combined with Fatou's lemma and the fact that $\sup_{t\in[0,T]} |\widehat{X}_t^{i,N,H_n}| \to \sup_{t\in[0,T]} |\widehat{X}_t^{i,N}|$ a.s. as $n \uparrow \infty$, we obtain that

$$\max_{i\in\{1,\dots,N\}} \mathbb{E}\left[\sup_{t\in[0,T]} |\widehat{X}_t^{i,N}|^p\right] \le \widetilde{C}_1,$$

which finishes the desired proof. □

Next, we are going to show that the moments of $\widehat{X}_t^{i,N}$ depend on t. For this, we need to introduce the following condition.

T4. There exists a positive constant L such that $|\sigma_\Delta(x,\mu)| \le |\sigma(x,\mu)|$ and $|c_\Delta(x,\mu)| \le |c(x,\mu)|$ for all $x \in \mathbb{R}^d$ and $\mu \in \mathcal{P}_2(\mathbb{R}^d)$;

T5. For some integer $p_0 \in [2,+\infty)$, there exist constants $\widetilde{L_3} > 0$, $\widetilde{\gamma}_1 \in \mathbb{R}$, $\widetilde{\gamma}_2 > 0$, $\widetilde{\eta} \ge 0$ such that for all $x \in \mathbb{R}^d$ and $\mu \in \mathcal{P}_2(\mathbb{R}^d)$,

$$|c_\Delta(x,\mu)| \le \widetilde{L_3}\left(1 + |x| + \mathcal{W}_2(\mu,\delta_0)\right), \tag{3.37}$$

and

$$\begin{aligned}&\langle x, b(x,\mu)\rangle + \frac{p_0-1}{2}|\sigma_\Delta(x,\mu)|^2 + |c_\Delta(x,\mu)|^2 \int_{\mathbb{R}_0^d}\Bigg[\frac{|z|^2}{2} + \frac{1}{\widetilde{L_3}^2} \\ &\times \left(\left(1+|z|(\widetilde{L_3}+\epsilon)\right)^{p_0-1} - 1 - |z|(\widetilde{L_3}+\epsilon)\right)\left(|z|\left(\frac{\widetilde{L_3}}{2}+\epsilon\right)+\epsilon\right)\Bigg]\nu(dz) \\ &\le \widetilde{\gamma}_1|x|^2 + \widetilde{\gamma}_2\mathcal{W}_2^2(\mu,\delta_0) + \widetilde{\eta},\end{aligned} \tag{3.38}$$

where $\epsilon = \frac{1}{2\sqrt{N}}\max\{3\widetilde{L_3}, 1\}$.

Remark 3.6.2. Observe that (3.38) of Condition **T5** yields to

$$\begin{aligned}&\langle x, b(x,\mu)\rangle + \frac{p-1}{2}|\sigma_\Delta(x,\mu)|^2 + |c_\Delta(x,\mu)|^2 \int_{\mathbb{R}_0^d}\Bigg[\frac{|z|^2}{2} + \frac{1}{\widetilde{L_3}^2} \\ &\times \left(\left(1+|z|(\widetilde{L_3}+\epsilon)\right)^{p-1} - 1 - |z|(\widetilde{L_3}+\epsilon)\right)\left(|z|\left(\frac{\widetilde{L_3}}{2}+\epsilon\right)+\epsilon\right)\Bigg]\nu(dz) \\ &\le \widetilde{\gamma}_1|x|^2 + \widetilde{\gamma}_2\mathcal{W}_2^2(\mu,\delta_0) + \widetilde{\eta}.\end{aligned}$$

for any $p \in [2, p_0]$, $x \in \mathbb{R}^d$ and $\mu \in \mathcal{P}_2(\mathbb{R}^d)$.

In the following, we state an estimate for L^2-norm of the approximate solution.

Lemma 3.6.3. *Assume* ***T1*–*T5*** *hold and* ***C5*** *holds with $p_0 = 2$. Then, there exists a positive constant $C = C(x_0, \widetilde{\gamma}_1, \widetilde{\gamma}_2, \widetilde{\eta}, L, \widetilde{L_3})$ which depends neither on Δ nor on t such that for any $t \geq 0$,*

$$\max_{i\in\{1,\dots,N\}} \left(\mathbb{E}\left[|\widehat{X}_t^{i,N}|^2\right] \vee \mathbb{E}\left[|\widehat{X}_{\underline{t}}^{i,N}|^2\right]\right) \leq \begin{cases} Ce^{2\widetilde{\gamma}t} & \text{if } \ \widetilde{\gamma} > 0, \\ C(1+t) & \text{if } \ \widetilde{\gamma} = 0, \\ C & \text{if } \ \widetilde{\gamma} < 0, \end{cases}$$

where $\widetilde{\gamma} = \widetilde{\gamma}_1 + \widetilde{\gamma}_2$.

Proof. Proceeding as in (3.12) by applying Itô's formula to $e^{-2\widetilde{\gamma}t}|\widehat{X}_t^{i,N}|^2$, we get that for any $i \in \{1, \dots, N\}$,

$$\begin{aligned}
&e^{-2\widetilde{\gamma}t}|\widehat{X}_t^{i,N}|^2 \\
&\leq |x_0|^2 + 2\int_0^t e^{-2\widetilde{\gamma}s}\Bigg(- \widetilde{\gamma}|\widehat{X}_s^{i,N}|^2 + \langle \widehat{X}_s^{i,N}, b(\widehat{X}_{\underline{s}}^{i,N}, \mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N})\rangle + \frac{1}{2}\left|\sigma_\Delta(\widehat{X}_{\underline{s}}^{i,N}, \mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N})\right|^2 \\
&\quad + \frac{1}{2}\left|c_\Delta(\widehat{X}_{\underline{s}}^{i,N}, \mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N})\right|^2 \int_{\mathbb{R}_0^d} |z|^2 \nu(dz)\Bigg) ds + 2\int_0^t e^{-2\widetilde{\gamma}s}\left\langle \widehat{X}_s^{i,N}, \sigma_\Delta(\widehat{X}_{\underline{s}}^{i,N}, \mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N}) dW_s^i \right\rangle \\
&\quad + \int_0^t \int_{\mathbb{R}_0^d} e^{-2\widetilde{\gamma}s}\left(2\left\langle \widehat{X}_{s-}^{i,N}, c_\Delta\left(\widehat{X}_{\underline{s-}}^{i,N}, \mu_{\underline{s-}}^{\widehat{\boldsymbol{X}}^N}\right) z\right\rangle + \left|c_\Delta\left(\widehat{X}_{\underline{s-}}^{i,N}, \mu_{\underline{s-}}^{\widehat{\boldsymbol{X}}^N}\right) z\right|^2 \right) \widetilde{N}^i(ds, dz).
\end{aligned} \tag{3.39}$$

First, using (3.31), we have

$$\begin{aligned}
&- \widetilde{\gamma}|\widehat{X}_s^{i,N}|^2 \\
&= -\widetilde{\gamma}|\widehat{X}_{\underline{s}}^{i,N}|^2 - 2\widetilde{\gamma}\left\langle \widehat{X}_{\underline{s}}^{i,N}, b\left(\widehat{X}_{\underline{s}}^{i,N}, \mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N}\right)(s - \underline{s}) + \sigma_\Delta\left(\widehat{X}_{\underline{s}}^{i,N}, \mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N}\right)(W_s^i - W_{\underline{s}}^i) \right. \\
&\quad \left. + c_\Delta\left(\widehat{X}_{\underline{s}}^{i,N}, \mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N}\right)(Z_s^i - Z_{\underline{s}}^i)\right\rangle - \widetilde{\gamma}\left| b\left(\widehat{X}_{\underline{s}}^{i,N}, \mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N}\right)(s - \underline{s}) \right. \\
&\quad \left. + \sigma_\Delta\left(\widehat{X}_{\underline{s}}^{i,N}, \mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N}\right)(W_s^i - W_{\underline{s}}^i) + c_\Delta\left(\widehat{X}_{\underline{s}}^{i,N}, \mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N}\right)(Z_s^i - Z_{\underline{s}}^i)\right|^2 \\
&\leq -\widetilde{\gamma}\left|\widehat{X}_{\underline{s}}^{i,N}\right|^2 + 2|\widetilde{\gamma}|\left|\widehat{X}_{\underline{s}}^{i,N}\right|\left|b\left(\widehat{X}_{\underline{s}}^{i,N}, \mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N}\right)\right|(s - \underline{s}) \\
&\quad - 2\widetilde{\gamma}\left\langle \widehat{X}_{\underline{s}}^{i,N}, \sigma_\Delta\left(\widehat{X}_{\underline{s}}^{i,N}, \mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N}\right)(W_s^i - W_{\underline{s}}^i) + c_\Delta\left(\widehat{X}_{\underline{s}}^{i,N}, \mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N}\right)(Z_s^i - Z_{\underline{s}}^i)\right\rangle \\
&\quad + 3|\widetilde{\gamma}|\Bigg(\left|b(\widehat{X}_{\underline{s}}^{i,N}, \mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N})\right|^2 (s - \underline{s})^2 + \left|\sigma_\Delta(\widehat{X}_{\underline{s}}^{i,N}, \mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N})\right|^2 \left|W_s^i - W_{\underline{s}}^i\right|^2 \\
&\quad + \left|c_\Delta(\widehat{X}_{\underline{s}}^{i,N}, \mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N})\right|^2 |Z_s^i - Z_{\underline{s}}^i|^2\Bigg).
\end{aligned}$$

Then, using **T4**, **C5**, (2.25), and (3.22), we get

$$\left|\widehat{X}^{i,N}_{\underline{s}}\right|\left|b(\widehat{X}^{i,N}_{\underline{s}},\mu^{\widehat{\boldsymbol{X}}^N}_{\underline{s}})\right|(s-\underline{s})\le C\Delta;$$
$$\left|b(\widehat{X}^{i,N}_{\underline{s}},\mu^{\widehat{\boldsymbol{X}}^N}_{\underline{s}})\right|^2(s-\underline{s})^2\le C\Delta;$$
$$\mathbb{E}\left[\left|\sigma_\Delta\left(\widehat{X}^{i,N}_{\underline{s}},\mu^{\widehat{\boldsymbol{X}}^N}_{\underline{s}}\right)\right|^2\left|W^i_s-W^i_{\underline{s}}\right|^2\big|\mathcal{F}_{\underline{s}}\right]\le C\Delta;$$
$$\mathbb{E}\left[\left|c_\Delta\left(\widehat{X}^{i,N}_{\underline{s}},\mu^{\widehat{\boldsymbol{X}}^N}_{\underline{s}}\right)\right|^2\left|Z^i_s-Z^i_{\underline{s}}\right|^2\big|\mathcal{F}_{\underline{s}}\right]\le C\Delta,$$

which yields that

$$\mathbb{E}\left[-\widetilde{\gamma}|\widehat{X}^{i,N}_s|^2\right]=\mathbb{E}\left[\mathbb{E}\left[-\widetilde{\gamma}|\widehat{X}^{i,N}_s|^2\big|\mathcal{F}_{\underline{s}}\right]\right]\le\mathbb{E}\left[-\widetilde{\gamma}|\widehat{X}^{i,N}_{\underline{s}}|^2\right]+C|\widetilde{\gamma}|\Delta. \tag{3.40}$$

Moreover, using again (3.31), we have

$$\begin{aligned}
&\left\langle\widehat{X}^{i,N}_s,b(\widehat{X}^{i,N}_{\underline{s}},\mu^{\widehat{\boldsymbol{X}}^N}_{\underline{s}})\right\rangle\\
&=\left\langle\widehat{X}^{i,N}_{\underline{s}},b(\widehat{X}^{i,N}_{\underline{s}},\mu^{\widehat{\boldsymbol{X}}^N}_{\underline{s}})\right\rangle+\left\langle\widehat{X}^{i,N}_s-\widehat{X}^{i,N}_{\underline{s}},b(\widehat{X}^{i,N}_{\underline{s}},\mu^{\widehat{\boldsymbol{X}}^N}_{\underline{s}})\right\rangle\\
&\le\left\langle\widehat{X}^{i,N}_{\underline{s}},b(\widehat{X}^{i,N}_{\underline{s}},\mu^{\widehat{\boldsymbol{X}}^N}_{\underline{s}})\right\rangle+\left|b(\widehat{X}^{i,N}_{\underline{s}},\mu^{\widehat{\boldsymbol{X}}^N}_{\underline{s}})\right|^2(s-\underline{s})\\
&\quad+\left\langle\sigma_\Delta\left(\widehat{X}^{i,N}_{\underline{s}},\mu^{\widehat{\boldsymbol{X}}^N}_{\underline{s}}\right)\left(W^i_s-W^i_{\underline{s}}\right)+c_\Delta\left(\widehat{X}^{i,N}_{\underline{s}},\mu^{\widehat{\boldsymbol{X}}^N}_{\underline{s}}\right)\left(Z^i_s-Z^i_{\underline{s}}\right),b(\widehat{X}^{i,N}_{\underline{s}},\mu^{\widehat{\boldsymbol{X}}^N}_{\underline{s}})\right\rangle,
\end{aligned}$$

which, combined with the fact that $|b(\widehat{X}^{i,N}_{\underline{s}},\mu^{\widehat{\boldsymbol{X}}^N}_{\underline{s}})|^2(s-\underline{s})\le\Delta$ due to (3.22), deduces that

$$\mathbb{E}\left[\left\langle\widehat{X}^{i,N}_s,b(\widehat{X}^{i,N}_{\underline{s}},\mu^{\widehat{\boldsymbol{X}}^N}_{\underline{s}})\right\rangle\right]\le\mathbb{E}\left[\left\langle\widehat{X}^{i,N}_{\underline{s}},b(\widehat{X}^{i,N}_{\underline{s}},\mu^{\widehat{\boldsymbol{X}}^N}_{\underline{s}})\right\rangle\right]+\Delta. \tag{3.41}$$

Thanks to Lemma 3.6.1, the stochastic integrals in (3.39) have zero expectation. Thus, using (3.39), (3.40), (3.41) and (3.38) of **T5** with $p_0=2$ and recall that $\widetilde{\gamma}=\widetilde{\gamma}_1+\widetilde{\gamma}_2$, we obtain that

$$\begin{aligned}
&\mathbb{E}\left[e^{-2\widetilde{\gamma}t}|\widehat{X}^{i,N}_t|^2\right]\\
&\le|x_0|^2+2\int_0^t e^{-2\widetilde{\gamma}s}\Bigg(\mathbb{E}\Bigg[-\widetilde{\gamma}|\widehat{X}^{i,N}_{\underline{s}}|^2+\left\langle\widehat{X}^{i,N}_{\underline{s}},b(\widehat{X}^{i,N}_{\underline{s}},\mu^{\widehat{\boldsymbol{X}}^N}_{\underline{s}})\right\rangle\\
&\quad+\frac{1}{2}\left|\sigma_\Delta(\widehat{X}^{i,N}_{\underline{s}},\mu^{\widehat{\boldsymbol{X}}^N}_{\underline{s}})\right|^2+\frac{1}{2}\left|c_\Delta(\widehat{X}^{i,N}_{\underline{s}},\mu^{\widehat{\boldsymbol{X}}^N}_{\underline{s}})\right|^2\int_{\mathbb{R}^d_0}|z|^2\nu(dz)\Bigg]+C\Bigg)ds\\
&\le|x_0|^2+2\int_0^t e^{-2\widetilde{\gamma}s}\Bigg(-\widetilde{\gamma}_2\mathbb{E}\left[|\widehat{X}^{i,N}_{\underline{s}}|^2\right]+\widetilde{\gamma}_2\mathbb{E}\left[\mathcal{W}_2^2(\mu^{\widehat{\boldsymbol{X}}^N}_{\underline{s}},\delta_0)\right]+\widetilde{\eta}+C\Bigg)ds
\end{aligned}$$

$$= |x_0|^2 + 2\int_0^t e^{-2\widetilde{\gamma}s}\left(-\widetilde{\gamma}_2\mathbb{E}\left[|\widehat{X}^{i,N}_{\underline{s}}|^2\right] + \widetilde{\gamma}_2\mathbb{E}\left[|\widehat{X}^{i,N}_{\underline{s}}|^2\right] + \widetilde{\eta} + C\right) ds$$
$$= |x_0|^2 + 2\left(\widetilde{\eta} + C\right)\int_0^t e^{-2\widetilde{\gamma}s} ds,$$

for some positive constant C, where we have used the equality $\mathbb{E}\left[\mathcal{W}_2^2(\mu^{\widehat{\boldsymbol{X}}^N}_{\underline{s}}, \delta_0)\right] = \mathbb{E}\left[|\widehat{X}^{i,N}_{\underline{s}}|^2\right]$. This yields to

$$\mathbb{E}\left[|\widehat{X}^{i,N}_t|^2\right] \leq \begin{cases} \left(|x_0|^2 + \dfrac{\widetilde{\eta} + C}{\widetilde{\gamma}}\right) e^{2\widetilde{\gamma}t} - \dfrac{\widetilde{\eta} + C}{\widetilde{\gamma}} & \text{if } \widetilde{\gamma} \neq 0, \\ |x_0|^2 + 2\left(\widetilde{\eta} + C\right) t & \text{if } \widetilde{\gamma} = 0. \end{cases} \tag{3.42}$$

Next, from (3.31), we have

$$\widehat{X}^{i,N}_{\underline{t}} = \widehat{X}^{i,N}_t - b\left(\widehat{X}^{i,N}_{\underline{t}}, \mu^{\widehat{\boldsymbol{X}}^N}_{\underline{t}}\right)(t - \underline{t}) - \sigma_\Delta\left(\widehat{X}^{i,N}_{\underline{t}}, \mu^{\widehat{\boldsymbol{X}}^N}_{\underline{t}}\right)\left(W^i_t - W^i_{\underline{t}}\right)$$
$$- c_\Delta\left(\widehat{X}^{i,N}_{\underline{t}}, \mu^{\widehat{\boldsymbol{X}}^N}_{\underline{t}}\right)\left(Z^i_t - Z^i_{\underline{t}}\right).$$

This, combined with **T4**, **C5**, (2.25) and (3.22), we get that for any $p > 1$,

$$\mathbb{E}\left[\left|\widehat{X}^{i,N}_{\underline{t}}\right|^p\right] \leq 4^{p-1}\Bigg(\mathbb{E}\left[\left|\widehat{X}^{i,N}_t\right|^p\right] + \mathbb{E}\left[\left|b(\widehat{X}^{i,N}_{\underline{t}}, \mu^{\widehat{\boldsymbol{X}}^N}_{\underline{t}})(t - \underline{t})\right|^p\right]$$
$$+ \mathbb{E}\left[\left|\sigma_\Delta(\widehat{X}^{i,N}_{\underline{t}}, \mu^{\widehat{\boldsymbol{X}}^N}_{\underline{t}})(W^i_t - W^i_{\underline{t}})\right|^p\right] + \mathbb{E}\left[\left|c_\Delta(\widehat{X}^{i,N}_{\underline{t}}, \mu^{\widehat{\boldsymbol{X}}^N}_{\underline{t}})(Z^i_t - Z^i_{\underline{t}})\right|^p\right]\Bigg)$$
$$\leq 4^{p-1}\left(\mathbb{E}\left[\left|\widehat{X}^{i,N}_t\right|^p\right] + C\Delta^p + C\Delta^{p/2} + C\Delta^{1\wedge p/2}\right). \tag{3.43}$$

Consequently, from (3.42) and (3.43) with $p = 2$, the result follows. □

To estimate L^p-norm of the approximate solution for $p > 2$, we need a series of preliminary lemmas.

Lemma 3.6.4. *Let p be a positive even integer. For any $a, b, c \in \mathbb{R}^d$, it holds that*

$$S = |a + c|^p - |b + c|^p - |a|^p + |b|^p$$
$$\leq \sum_{j=1}^{p/2}\sum_{k=0}^{j}\binom{p/2}{j}\binom{j}{k} 2^{j-k}|c|^{j+k}\Bigg[2^{p-2j-1}(|a - b|^{p-2j} + |b|^{p-2j})$$
$$\times \sum_{\ell=1}^{j-k}\binom{j-k}{\ell}|b|^{j-k-\ell}|a - b|^{\ell} + \sum_{\ell=1}^{p-2j}\binom{p-2j}{\ell}|b|^{p-j-k-\ell}|a - b|^{\ell}\Bigg]. \tag{3.44}$$

Proof. We first note that $S=\Big(|a|^2+2\langle a,c\rangle+|c|^2\Big)^{p/2}-|a|^p-\Big(|b|^2+2\langle b,c\rangle+|c|^2\Big)^{p/2}+|b|^p$. Using the binomial theorem, we have

$$S=\sum_{j=1}^{p/2}\sum_{k=0}^{j}\binom{p/2}{j}\binom{j}{k}2^{j-k}|c|^{2k}\Big(|a|^{p-2j}\langle a,c\rangle^{j-k}-|b|^{p-2j}\langle b,c\rangle^{j-k}\Big).$$

Next, we write

$$\begin{aligned}|a|^{p-2j}\langle a,c\rangle^{j-k}-|b|^{p-2j}\langle b,c\rangle^{j-k}&=|a|^{p-2j}\Big((\langle b,c\rangle+\langle a-b,c\rangle)^{j-k}-\langle b,c\rangle^{j-k}\Big)\\&\quad+\Big((|b|+(|a|-|b|))^{p-2j}-|b|^{p-2j}\Big)\langle b,c\rangle^{j-k}.\end{aligned}$$

Using the binomial theorem, the estimates $|a|^{p-2j}\leq 2^{p-2j-1}(|a-b|^{p-2j}+|b|^{p-2j})$, $|\langle a-b,c\rangle|\leq|a-b||c|$, and $|\langle b,c\rangle|\leq|b||c|$, we obtain the desired result. □

Lemma 3.6.5. *Assume Conditions **T1**–**T5** and **C5** hold. Then, for any even integer $p\in(0;p_0]$, there exists a positive constant C_p such that for any $s>0$, $i\in\{1,\dots,N\}$ and $\lambda\in\mathbb{R}$,*

a) $\mathbb{E}\Big[-\lambda|\widehat{X}_s^{i,N}|^p\Big|\mathcal{F}_{\underline{s}}\Big]\leq-\lambda|\widehat{X}_{\underline{s}}^{i,N}|^p+C_p|\lambda|\sum_{j=0}^{p-2}|\widehat{X}_{\underline{s}}^{i,N}|^j.$

b) $\mathbb{E}\Big[|\widehat{X}_s^{i,N}|^{p-2}\Big\langle\widehat{X}_s^{i,N},b(\widehat{X}_{\underline{s}}^{i,N},\mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N})\Big\rangle\Big|\mathcal{F}_{\underline{s}}\Big]\leq|\widehat{X}_{\underline{s}}^{i,N}|^{p-2}\Big\langle\widehat{X}_{\underline{s}}^{i,N},b(\widehat{X}_{\underline{s}}^{i,N},\mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N})\Big\rangle+C_p\sum_{j=0}^{p-2}|\widehat{X}_{\underline{s}}^{i,N}|^j.$

c) $\mathbb{E}\Big[|\widehat{X}_s^{i,N}|^{p-4}|(\widehat{X}_s^{i,N})^{\mathsf{T}}\sigma_\Delta(\widehat{X}_{\underline{s}}^{i,N},\mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N})|^2\Big|\mathcal{F}_{\underline{s}}\Big]\leq|\widehat{X}_{\underline{s}}^{i,N}|^{p-2}|\sigma_\Delta(\widehat{X}_{\underline{s}}^{i,N},\mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N})|^2+C_p\sum_{j=0}^{p-3}|\widehat{X}_{\underline{s}}^{i,N}|^j.$

Proof. First, by using **T4**, (3.22), Burkholder-Davis-Gundy's inequality, (2.25) and **C5**, we get that for all $2\leq j\leq p$,

$$\begin{aligned}&\max\Big\{|\widehat{X}_{\underline{s}}^{i,N}|\Big|b\Big(\widehat{X}_{\underline{s}}^{i,N},\mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N}\Big)\Big||s-\underline{s}|\,;\mathbb{E}\Big[|\widehat{X}_{\underline{s}}^{i,N}|\Big|\sigma_\Delta\Big(\widehat{X}_{\underline{s}}^{i,N},\mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N}\Big)\Big|\Big|W_s^i-W_{\underline{s}}^i\Big|\Big|\mathcal{F}_{\underline{s}}\Big]\,;\\&\quad\mathbb{E}\Big[|\widehat{X}_{\underline{s}}^{i,N}|\Big|c_\Delta\Big(\widehat{X}_{\underline{s}}^{i,N},\mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N}\Big)\Big|\Big|Z_s^i-Z_{\underline{s}}^i\Big|\Big|\mathcal{F}_{\underline{s}}\Big]\Big\}\leq C\sqrt{\Delta},\\&\max\Big\{\Big|b\Big(\widehat{X}_{\underline{s}}^{i,N},\mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N}\Big)\Big|^j|s-\underline{s}|^j\,;\ \mathbb{E}\Big[\Big|\sigma_\Delta\Big(\widehat{X}_{\underline{s}}^{i,N},\mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N}\Big)\Big|^j\Big|W_s^i-W_{\underline{s}}^i\Big|^j\Big|\mathcal{F}_{\underline{s}}\Big]\,;\\&\quad\mathbb{E}\Big[\Big|c_\Delta\Big(\widehat{X}_{\underline{s}}^{i,N},\mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N}\Big)\Big|^j\Big|Z_s^i-Z_{\underline{s}}^i\Big|^j\Big|\mathcal{F}_{\underline{s}}\Big]\Big\}\leq C\Delta.\end{aligned}\tag{3.45}$$

a) Using the binomial theorem and (3.31), we get that

$$-\lambda|\widehat{X}_s^{i,N}|^p$$

$$
\begin{aligned}
&= -\lambda|\widehat{X}^{i,N}_{\underline{s}}|^p - \lambda \sum_{j=1}^{p} \binom{p}{j} |\widehat{X}^{i,N}_{\underline{s}}|^{p-j} \left(|\widehat{X}^{i,N}_{s}| - |\widehat{X}^{i,N}_{\underline{s}}|\right)^j \\
&\leq -\lambda|\widehat{X}^{i,N}_{\underline{s}}|^p + |\lambda| p |\widehat{X}^{i,N}_{\underline{s}}|^{p-1} \left|\widehat{X}^{i,N}_{s} - \widehat{X}^{i,N}_{\underline{s}}\right| + |\lambda| \sum_{j=2}^{p} \binom{p}{j} |\widehat{X}^{i,N}_{\underline{s}}|^{p-j} \left|\widehat{X}^{i,N}_{s} - \widehat{X}^{i,N}_{\underline{s}}\right|^j \\
&\leq -\lambda|\widehat{X}^{i,N}_{\underline{s}}|^p + |\lambda| p |\widehat{X}^{i,N}_{\underline{s}}|^{p-2} \Bigg[|\widehat{X}^{i,N}_{\underline{s}}| \left| b\left(\widehat{X}^{i,N}_{\underline{s}}, \mu^{\widehat{\boldsymbol{X}}^N}_{\underline{s}}\right)\right| |s - \underline{s}| \\
&\quad + |\widehat{X}^{i,N}_{\underline{s}}| \left| \sigma_\Delta\left(\widehat{X}^{i,N}_{\underline{s}}, \mu^{\widehat{\boldsymbol{X}}^N}_{\underline{s}}\right)\right| |W^i_s - W^i_{\underline{s}}| + |\widehat{X}^{i,N}_{\underline{s}}| \left| c_\Delta\left(\widehat{X}^{i,N}_{\underline{s}}, \mu^{\widehat{\boldsymbol{X}}^N}_{\underline{s}}\right)\right| |Z^i_s - Z^i_{\underline{s}}| \Bigg] \\
&\quad + |\lambda| \sum_{j=2}^{p} 3^{j-1} \binom{p}{j} |\widehat{X}^{i,N}_{\underline{s}}|^{p-j} \Bigg[\left| b\left(\widehat{X}^{i,N}_{\underline{s}}, \mu^{\widehat{\boldsymbol{X}}^N}_{\underline{s}}\right)\right|^j |s - \underline{s}|^j \\
&\quad + \left| \sigma_\Delta\left(\widehat{X}^{i,N}_{\underline{s}}, \mu^{\widehat{\boldsymbol{X}}^N}_{\underline{s}}\right)\right|^j \left|W^i_s - W^i_{\underline{s}}\right|^j + \left| c_\Delta\left(\widehat{X}^{i,N}_{\underline{s}}, \mu^{\widehat{\boldsymbol{X}}^N}_{\underline{s}}\right)\right|^j \left|Z^i_s - Z^i_{\underline{s}}\right|^j \Bigg].
\end{aligned}
$$

Using (3.45), we conclude part a) of Lemma 3.6.5.

b) Using a similar computation as in part a), we obtain

$$
\begin{aligned}
&|\widehat{X}^{i,N}_{s}|^{p-2} \left\langle \widehat{X}^{i,N}_{s}, b(\widehat{X}^{i,N}_{\underline{s}}, \mu^{\widehat{\boldsymbol{X}}^N}_{\underline{s}}) \right\rangle \\
&\leq |\widehat{X}^{i,N}_{\underline{s}}|^{p-2} \left\langle \widehat{X}^{i,N}_{\underline{s}}, b(\widehat{X}^{i,N}_{\underline{s}}, \mu^{\widehat{\boldsymbol{X}}^N}_{\underline{s}}) \right\rangle + |\widehat{X}^{i,N}_{\underline{s}}|^{p-2} |b(\widehat{X}^{i,N}_{\underline{s}}, \mu^{\widehat{\boldsymbol{X}}^N}_{\underline{s}})|^2 (s - \underline{s}) \\
&\quad + |\widehat{X}^{i,N}_{\underline{s}}|^{p-2} \Big\langle \sigma_\Delta\left(\widehat{X}^{i,N}_{\underline{s}}, \mu^{\widehat{\boldsymbol{X}}^N}_{\underline{s}}\right) \left(W^i_s - W^i_{\underline{s}}\right) + c_\Delta\left(\widehat{X}^{i,N}_{\underline{s}}, \mu^{\widehat{\boldsymbol{X}}^N}_{\underline{s}}\right) \left(Z^i_s - Z^i_{\underline{s}}\right), \\
&\quad b(\widehat{X}^{i,N}_{\underline{s}}, \mu^{\widehat{\boldsymbol{X}}^N}_{\underline{s}}) \Big\rangle + \sum_{j=1}^{p-2} \binom{p-2}{j} |\widehat{X}^{i,N}_{\underline{s}}|^{p-1-j} 3^{j-1} |b(\widehat{X}^{i,N}_{\underline{s}}, \mu^{\widehat{\boldsymbol{X}}^N}_{\underline{s}})| \\
&\quad \times \Bigg[\left| b\left(\widehat{X}^{i,N}_{\underline{s}}, \mu^{\widehat{\boldsymbol{X}}^N}_{\underline{s}}\right)\right|^j |s - \underline{s}|^j + \left| \sigma_\Delta\left(\widehat{X}^{i,N}_{\underline{s}}, \mu^{\widehat{\boldsymbol{X}}^N}_{\underline{s}}\right)\right|^j \left|W^i_s - W^i_{\underline{s}}\right|^j \\
&\quad + \left| c_\Delta\left(\widehat{X}^{i,N}_{\underline{s}}, \mu^{\widehat{\boldsymbol{X}}^N}_{\underline{s}}\right)\right|^j \left|Z^i_s - Z^i_{\underline{s}}\right|^j \Bigg] + \sum_{j=1}^{p-2} \binom{p-2}{j} |\widehat{X}^{i,N}_{\underline{s}}|^{p-2-j} 3^j |b(\widehat{X}^{i,N}_{\underline{s}}, \mu^{\widehat{\boldsymbol{X}}^N}_{\underline{s}})| \\
&\quad \times \Bigg[\left| b\left(\widehat{X}^{i,N}_{\underline{s}}, \mu^{\widehat{\boldsymbol{X}}^N}_{\underline{s}}\right)\right|^{j+1} |s - \underline{s}|^{j+1} + \left| \sigma_\Delta\left(\widehat{X}^{i,N}_{\underline{s}}, \mu^{\widehat{\boldsymbol{X}}^N}_{\underline{s}}\right)\right|^{j+1} \left|W^i_s - W^i_{\underline{s}}\right|^{j+1} \\
&\quad + \left| c_\Delta\left(\widehat{X}^{i,N}_{\underline{s}}, \mu^{\widehat{\boldsymbol{X}}^N}_{\underline{s}}\right)\right|^{j+1} \left|Z^i_s - Z^i_{\underline{s}}\right|^{j+1} \Bigg].
\end{aligned}
$$

Using again (3.45), we conclude part b) of Lemma 3.6.5.

c) Using the binomial theorem and (3.31), we have

$$
\begin{aligned}
&|\widehat{X}_s^{i,N}|^{p-2}|\sigma_\Delta(\widehat{X}_{\underline{s}}^{i,N},\mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N})|^2\\
&\leq |\widehat{X}_{\underline{s}}^{i,N}|^{p-2}|\sigma_\Delta(\widehat{X}_{\underline{s}}^{i,N},\mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N})|^2+\sum_{j=1}^{p-2}\binom{p-2}{j}|\widehat{X}_{\underline{s}}^{i,N}|^{p-2-j}|\sigma_\Delta(\widehat{X}_{\underline{s}}^{i,N},\mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N})|^2 3^{j-1}\\
&\times\Bigg[\left|b\left(\widehat{X}_{\underline{s}}^{i,N},\mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N}\right)\right|^j|s-\underline{s}|^j+\left|\sigma_\Delta\left(\widehat{X}_{\underline{s}}^{i,N},\mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N}\right)\right|^j\left|W_s^i-W_{\underline{s}}^i\right|^j\\
&+\left|c_\Delta\left(\widehat{X}_{\underline{s}}^{i,N},\mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N}\right)\right|^j\left|Z_s^i-Z_{\underline{s}}^i\right|^j\Bigg].
\end{aligned}
$$

Then, using the estimates (3.45), we get that

$$
\begin{aligned}
&\mathbb{E}\left[|\widehat{X}_s^{i,N}|^{p-4}|(\widehat{X}_s^{i,N})^\mathsf{T}\sigma_\Delta(\widehat{X}_{\underline{s}}^{i,N},\mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N})|^2\Big|\mathcal{F}_{\underline{s}}\right]\\
&\leq \mathbb{E}\left[|\widehat{X}_s^{i,N}|^{p-2}|\sigma_\Delta(\widehat{X}_{\underline{s}}^{i,N},\mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N})|^2\Big|\mathcal{F}_{\underline{s}}\right]\\
&\leq |\widehat{X}_{\underline{s}}^{i,N}|^{p-2}|\sigma_\Delta(\widehat{X}_{\underline{s}}^{i,N},\mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N})|^2+C_p\sum_{j=0}^{p-3}|\widehat{X}_{\underline{s}}^{i,N}|^j. \qquad (3.46)
\end{aligned}
$$

We conclude part c) of Lemma 3.6.5. □

Lemma 3.6.6. *Assume Conditions **T1**–**T5** and **C5** hold. Then, for any even integer $p\in(0,p_0]$, $s>0$, and $z\in\mathbb{R}^d$, it holds that*

$$
\begin{aligned}
&\mathbb{E}\Bigg[\left(\left|\widehat{X}_s^{i,N}+c_\Delta(\widehat{X}_{\underline{s}}^{i,N},\mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N})z\right|^p-\left|\widehat{X}_{\underline{s}}^{i,N}+c_\Delta(\widehat{X}_{\underline{s}}^{i,N},\mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N})z\right|^p\right)\\
&-\left(|\widehat{X}_s^{i,N}|^p-|\widehat{X}_{\underline{s}}^{i,N}|^p\right)\Bigg|\mathcal{F}_{\underline{s}}\Bigg]\leq \widehat{Q}_{p-2}\left(\left|\widehat{X}_{\underline{s}}^{i,N}\right|,|z|\right)+\sum_{j=1}^{p/2}\sum_{k=0}^{j}C_{j,k}|z|^{j+k}\mathcal{W}_2^{j+k}(\mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N},\delta_0)\\
&\times\Bigg[\left|\widehat{X}_{\underline{s}}^{i,N}\right|^{j-k-2}+\left|\widehat{X}_{\underline{s}}^{i,N}\right|^{p-j-k-2}+\sum_{\ell=2}^{j-k}\left(\left|\widehat{X}_{\underline{s}}^{i,N}\right|^{j-k-\ell}+\left|\widehat{X}_{\underline{s}}^{i,N}\right|^{p-j-k-\ell}\right)\\
&+\sum_{\ell=2}^{p-2j}\left|\widehat{X}_{\underline{s}}^{i,N}\right|^{p-j-k-\ell}\Bigg], \qquad (3.47)
\end{aligned}
$$

where $(C_{j,k})$ are some positive constants , $\widehat{Q}_{p-2}(|\widehat{X}_{\underline{s}}^{i,N}|,|z|)$ is a polynomial in $|\widehat{X}_{\underline{s}}^{i,N}|$ of degree $p-2$.

Proof. By using Lemma 3.6.4, we have

$$
\left(\left|\widehat{X}_s^{i,N}+c_\Delta(\widehat{X}_{\underline{s}}^{i,N},\mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N})z\right|^p-\left|\widehat{X}_{\underline{s}}^{i,N}+c_\Delta(\widehat{X}_{\underline{s}}^{i,N},\mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N})z\right|^p\right)-\left(|\widehat{X}_s^{i,N}|^p-|\widehat{X}_{\underline{s}}^{i,N}|^p\right)
$$

$$
\le \sum_{j=1}^{p/2}\sum_{k=0}^{j}\binom{p/2}{j}\binom{j}{k}2^{j-k}\left|c_\Delta(\widehat{X}_{\underline{s}}^{i,N},\mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N})z\right|^{j+k}\Bigg[2^{p-2j-1}\sum_{\ell=1}^{j-k}\binom{j-k}{\ell}\left|\widehat{X}_{\underline{s}}^{i,N}\right|^{j-k-\ell}
$$

$$
\times\left|\widehat{X}_{s}^{i,N}-\widehat{X}_{\underline{s}}^{i,N}\right|^{p-2j+\ell}+2^{p-2j-1}\sum_{\ell=1}^{j-k}\binom{j-k}{\ell}\left|\widehat{X}_{\underline{s}}^{i,N}\right|^{p-j-k-\ell}\left|\widehat{X}_{s}^{i,N}-\widehat{X}_{\underline{s}}^{i,N}\right|^{\ell}
$$

$$
+\sum_{\ell=1}^{p-2j}\binom{p-2j}{\ell}\left|\widehat{X}_{\underline{s}}^{i,N}\right|^{p-j-k-\ell}\left|\widehat{X}_{s}^{i,N}-\widehat{X}_{\underline{s}}^{i,N}\right|^{\ell}\Bigg].
$$

Then, using (3.37) of **T5** and (3.31), we get

$$
\left(\left|\widehat{X}_{s}^{i,N}+c_\Delta(\widehat{X}_{\underline{s}}^{i,N},\mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N})z\right|^{p}-\left|\widehat{X}_{\underline{s}}^{i,N}+c_\Delta(\widehat{X}_{\underline{s}}^{i,N},\mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N})z\right|^{p}\right)-\left(|\widehat{X}_{s}^{i,N}|^p-|\widehat{X}_{\underline{s}}^{i,N}|^p\right)
$$

$$
\le \sum_{j=1}^{p/2}\sum_{k=0}^{j}\binom{p/2}{j}\binom{j}{k}2^{j-k}|z|^{j+k}(\widetilde{L_3})^{j+k}3^{j+k-1}\left(1+\left|\widehat{X}_{\underline{s}}^{i,N}\right|^{j+k}+\mathcal{W}_2^{j+k}(\mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N},\delta_0)\right)
$$

$$
\times\Bigg[2^{p-2j-1}(j-k)\left|\widehat{X}_{\underline{s}}^{i,N}\right|^{j-k-2}3^{p-2j}\Bigg(\left|\widehat{X}_{\underline{s}}^{i,N}\right|\left|b\left(\widehat{X}_{\underline{s}}^{i,N},\mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N}\right)\right|^{p-2j+1}|s-\underline{s}|^{p-2j+1}
$$

$$
+\left|\widehat{X}_{\underline{s}}^{i,N}\right|\left|\sigma_\Delta\left(\widehat{X}_{\underline{s}}^{i,N},\mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N}\right)\right|^{p-2j+1}\left|W_s^i-W_{\underline{s}}^i\right|^{p-2j+1}+\left|\widehat{X}_{\underline{s}}^{i,N}\right|\left|c_\Delta\left(\widehat{X}_{\underline{s}}^{i,N},\mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N}\right)\right|^{p-2j+1}
$$

$$
\times\left|Z_s^i-Z_{\underline{s}}^i\right|^{p-2j+1}\Bigg)+\left(2^{p-2j-1}\binom{j-k}{1}+\binom{p-2j}{1}\right)\left|\widehat{X}_{\underline{s}}^{i,N}\right|^{p-j-k-2}
$$

$$
\times\Bigg(|\widehat{X}_{\underline{s}}^{i,N}||b(\widehat{X}_{\underline{s}}^{i,N},\mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N})|(s-\underline{s})+|\widehat{X}_{\underline{s}}^{i,N}|\left|\sigma_\Delta\left(\widehat{X}_{\underline{s}}^{i,N},\mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N}\right)\right|\left|W_s^i-W_{\underline{s}}^i\right|
$$

$$
+|\widehat{X}_{\underline{s}}^{i,N}|\left|c_\Delta\left(\widehat{X}_{\underline{s}}^{i,N},\mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N}\right)\right|\left|Z_s^i-Z_{\underline{s}}^i\right|\Bigg)\Bigg]+\sum_{j=1}^{p/2}\sum_{k=0}^{j}\binom{p/2}{j}\binom{j}{k}2^{j-k}|z|^{j+k}
$$

$$
\times(\widetilde{L_3})^{j+k}3^{j+k-1}\left(1+|\widehat{X}_{\underline{s}}^{i,N}|^{j+k}+\mathcal{W}_2^{j+k}(\mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N},\delta_0)\right)\Bigg[]2^{p-2j-1}\sum_{\ell=2}^{j-k}\binom{j-k}{\ell}
$$

$$
\times\left|\widehat{X}_{\underline{s}}^{i,N}\right|^{j-k-\ell}3^{p-2j+\ell-1}\Bigg(\left|b\left(\widehat{X}_{\underline{s}}^{i,N},\mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N}\right)\right|^{p-2j+\ell}|s-\underline{s}|^{p-2j+\ell}
$$

$$
+\left|\sigma_\Delta\left(\widehat{X}_{\underline{s}}^{i,N},\mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N}\right)\right|^{p-2j+\ell}\left|W_s^i-W_{\underline{s}}^i\right|^{p-2j+\ell}+\left|c_\Delta\left(\widehat{X}_{\underline{s}}^{i,N},\mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N}\right)\right|^{p-2j+\ell}
$$

$$
\times\left|Z_s^i-Z_{\underline{s}}^i\right|^{p-2j+\ell}\Bigg)+2^{p-2j-1}\sum_{\ell=2}^{j-k}\binom{j-k}{\ell}\left|\widehat{X}_{\underline{s}}^{i,N}\right|^{p-j-k-\ell}3^{\ell-1}\Bigg(\left|b\left(\widehat{X}_{\underline{s}}^{i,N},\mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N}\right)\right|^{\ell}
$$

$$
\times|s-\underline{s}|^{\ell}+\left|\sigma_\Delta\left(\widehat{X}_{\underline{s}}^{i,N},\mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N}\right)\right|^{\ell}\left|W_s^i-W_{\underline{s}}^i\right|^{\ell}+\left|c_\Delta\left(\widehat{X}_{\underline{s}}^{i,N},\mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N}\right)\right|^{\ell}\left|Z_s^i-Z_{\underline{s}}^i\right|^{\ell}\Bigg)
$$

$$+\sum_{\ell=2}^{p-2j}\binom{p-2j}{\ell}3^{\ell-1}\left|\widehat{X}_{\underline{s}}^{i,N}\right|^{p-j-k-\ell}\left(\left|b\left(\widehat{X}_{\underline{s}}^{i,N},\mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N}\right)\right|^{\ell}|s-\underline{s}|^{\ell}+\left|\sigma_\Delta\left(\widehat{X}_{\underline{s}}^{i,N},\mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N}\right)\right|^{\ell}\right.$$
$$\left.\left.\times\left|W_s^i-W_{\underline{s}}^i\right|^{\ell}+\left|c_\Delta\left(\widehat{X}_{\underline{s}}^{i,N},\mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N}\right)\right|^{\ell}\left|Z_s^i-Z_{\underline{s}}^i\right|^{\ell}\right)\right].$$

By applying the estimates (3.45) and condition **T4**, we obtain the desired result. □

Lemma 3.6.7. *Assume Conditions* ***T1***–***T5*** *and* ***C5*** *hold. Then, for any even integer* $p\in(0,p_0]$, $s>0$, *and* $z\in\mathbb{R}^d$, *it holds*

$$\left|\widehat{X}_{\underline{s}}^{i,N}+c_\Delta(\widehat{X}_{\underline{s}}^{i,N},\mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N})z\right|^p-|\widehat{X}_{\underline{s}}^{i,N}|^p-p|\widehat{X}_{\underline{s}}^{i,N}|^{p-2}\left\langle\widehat{X}_{\underline{s}}^{i,N},c_\Delta(\widehat{X}_{\underline{s}}^{i,N},\mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N})z\right\rangle$$
$$\leq\left|c_\Delta(\widehat{X}_{\underline{s}}^{i,N},\mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N})\right|^2|\widehat{X}_{\underline{s}}^{i,N}|^{p-2}p\left(\frac{|z|^2}{2}+\widetilde{L_3}^{-2}\left(\left(1+|z|(\widetilde{L_3}+\epsilon)\right)^{p-1}-|z|(\widetilde{L_3}+\epsilon)-1\right)\right.$$
$$\left.\times\left(|z|\left(\frac{\widetilde{L_3}}{2}+\epsilon\right)+\epsilon\right)\right)+\sum_{k=2}^{p/2}\sum_{\ell=0}^{k}\binom{p/2}{k}|z|^{2k-\ell}Q_p\left(p-2,2k-\ell,|\widehat{X}_{\underline{s}}^{i,N}|,1+\widehat{U}_{\underline{s}}^{i,N}\right),\tag{3.48}$$

where $\epsilon=\frac{1}{2\sqrt{N}}\max\{3\widetilde{L_3},1\}$, $\widehat{U}_{\underline{s}}^{i,N}:=\frac{1}{\sqrt{N}}\sum_{j=1;j\neq i}^{N}\left|\widehat{X}_{\underline{s}}^{j,N}\right|$, *and*

$$Q_p\left(p-2,2k-\ell,|\widehat{X}_{\underline{s}}^{i,N}|,1+\widehat{U}_{\underline{s}}^{i,N}\right)$$
$$:=2\widetilde{L_3}^2\left(\left(1+\frac{1}{\sqrt{N}}\right)^2|\widehat{X}_{\underline{s}}^{i,N}|^2+\left(1+\widehat{U}_{\underline{s}}^{i,N}\right)^2\right)\binom{k}{\ell}2^{\ell}\widetilde{L_3}^{2k-\ell-2}$$
$$\times\left[\left(k-\frac{\ell}{2}-1\right)(1+\frac{1}{\sqrt{N}})^{2k-\ell-3}\left(1+\widehat{U}_{\underline{s}}^{i,N}\right)^2|\widehat{X}_{\underline{s}}^{i,N}|^{p-4}\right.$$
$$\left.+\sum_{m=2}^{2k-\ell-2}\binom{2k-\ell-2}{m}(1+\frac{1}{\sqrt{N}})^{2k-\ell-2-m}\left(1+\widehat{U}_{\underline{s}}^{i,N}\right)^m|\widehat{X}_{\underline{s}}^{i,N}|^{p-2-m}\right].$$

Proof. Proceeding in the same way as in (3.7), we get

$$\left|\widehat{X}_{\underline{s}}^{i,N}+c_\Delta(\widehat{X}_{\underline{s}}^{i,N},\mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N})z\right|^p-|\widehat{X}_{\underline{s}}^{i,N}|^p-p|\widehat{X}_{\underline{s}}^{i,N}|^{p-2}\left\langle\widehat{X}_{\underline{s}}^{i,N},c_\Delta(\widehat{X}_{\underline{s}}^{i,N},\mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N})z\right\rangle$$
$$=\frac{p}{2}\left|\widehat{X}_{\underline{s}}^{i,N}\right|^{p-2}\left|c_\Delta(\widehat{X}_{\underline{s}}^{i,N},\mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N})z\right|^2+\sum_{k=2}^{p/2}\binom{p/2}{k}|\widehat{X}_{\underline{s}}^{i,N}|^{p-2k}\sum_{\ell=0}^{k}\binom{k}{\ell}$$
$$\times\left|c_\Delta(\widehat{X}_{\underline{s}}^{i,N},\mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N})z\right|^{2k-2\ell}2^{\ell}\left(\left\langle\widehat{X}_{\underline{s}}^{i,N},c_\Delta(\widehat{X}_{\underline{s}}^{i,N},\mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N})z\right\rangle\right)^{\ell}$$
$$\leq\frac{p}{2}\left|\widehat{X}_{\underline{s}}^{i,N}\right|^{p-2}\left|c_\Delta(\widehat{X}_{\underline{s}}^{i,N},\mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N})z\right|^2+\sum_{k=2}^{p/2}\sum_{\ell=0}^{k}\binom{p/2}{k}\binom{k}{\ell}2^{\ell}|\widehat{X}_{\underline{s}}^{i,N}|^{p-2k+\ell}$$

$$\times \left| c_\Delta(\widehat{X}^{i,N}_{\underline{s}}, \mu^{\widehat{\boldsymbol{X}}^N}_{\underline{s}}) \right|^{2k-\ell} |z|^{2k-\ell}.$$

It follows from estimate (3.37) of **T5** and the estimate $\mathcal{W}_2(\mu^{\widehat{\boldsymbol{X}}^N}_{\underline{s}}, \delta_0) \le \frac{1}{\sqrt{N}} \sum_{j=1}^N |\widehat{X}^{j,N}_{\underline{s}}|$ that

$$\begin{aligned}
&\left| c_\Delta(\widehat{X}^{i,N}_{\underline{s}}, \mu^{\widehat{\boldsymbol{X}}^N}_{\underline{s}}) \right|^{2k-\ell-2} \le \widetilde{L_3}^{2k-\ell-2} \left(1 + |\widehat{X}^{i,N}_{\underline{s}}| + \mathcal{W}_2(\mu^{\widehat{\boldsymbol{X}}^N}_{\underline{s}}, \delta_0) \right)^{2k-\ell-2} \\
&\le \widetilde{L_3}^{2k-\ell-2} \left(1 + |\widehat{X}^{i,N}_{\underline{s}}| + \frac{1}{\sqrt{N}} \sum_{j=1}^N \left| \widehat{X}^{j,N}_{\underline{s}} \right| \right)^{2k-\ell-2} \\
&= \widetilde{L_3}^{2k-\ell-2} \left((1 + \frac{1}{\sqrt{N}}) |\widehat{X}^{i,N}_{\underline{s}}| + 1 + \frac{1}{\sqrt{N}} \sum_{j=1; j\neq i}^N \left| \widehat{X}^{j,N}_{\underline{s}} \right| \right)^{2k-\ell-2} \\
&= \widetilde{L_3}^{2k-\ell-2} \Bigg[(1 + \frac{1}{\sqrt{N}})^{2k-\ell-2} |\widehat{X}^{i,N}_{\underline{s}}|^{2k-\ell-2} + (2k-\ell-2) \left(1 + \widehat{U}^{i,N}_{\underline{s}} \right) \\
&\quad \times (1 + \frac{1}{\sqrt{N}})^{2k-\ell-3} |\widehat{X}^{i,N}_{\underline{s}}|^{2k-\ell-3} + \sum_{m=2}^{2k-\ell-2} \binom{2k-\ell-2}{m} \left(1 + \widehat{U}^{i,N}_{\underline{s}} \right)^m \\
&\quad \times (1 + \frac{1}{\sqrt{N}})^{2k-\ell-2-m} |\widehat{X}^{i,N}_{\underline{s}}|^{2k-\ell-2-m} \Bigg].
\end{aligned} \tag{3.49}$$

Using the estimate $\left(1 + \widehat{U}^{i,N}_{\underline{s}} \right) |\widehat{X}^{i,N}_{\underline{s}}|^{2k-\ell-3} \le \frac{1}{2} \Big(\left(1 + \widehat{U}^{i,N}_{\underline{s}} \right)^2 |\widehat{X}^{i,N}_{\underline{s}}|^{2k-\ell-4} + |\widehat{X}^{i,N}_{\underline{s}}|^{2k-\ell-2} \Big)$, we get

$$\begin{aligned}
&\sum_{\ell=0}^k \binom{k}{\ell} 2^\ell |\widehat{X}^{i,N}_{\underline{s}}|^{p-2k+\ell} \left| c_\Delta(\widehat{X}^{i,N}_{\underline{s}}, \mu^{\widehat{\boldsymbol{X}}^N}_{\underline{s}}) \right|^{2k-\ell} |z|^{2k-\ell} \\
&\le \left| c_\Delta(\widehat{X}^{i,N}_{\underline{s}}, \mu^{\widehat{\boldsymbol{X}}^N}_{\underline{s}}) \right|^2 |\widehat{X}^{i,N}_{\underline{s}}|^{p-2} \sum_{\ell=0}^k \binom{k}{\ell} 2^\ell |z|^{2k-\ell} \widetilde{L_3}^{2k-\ell-2} \left(1 + \frac{1}{\sqrt{N}} \right)^{2k-\ell-3} \\
&\quad \times \left(\frac{1}{\sqrt{N}} + k - \frac{\ell}{2} \right) + \left| c_\Delta(\widehat{X}^{i,N}_{\underline{s}}, \mu^{\widehat{\boldsymbol{X}}^N}_{\underline{s}}) \right|^2 \sum_{\ell=0}^k \binom{k}{\ell} 2^\ell |z|^{2k-\ell} \widetilde{L_3}^{2k-\ell-2} \\
&\quad \times \Bigg[\left(k - \frac{\ell}{2} - 1 \right) (1 + \frac{1}{\sqrt{N}})^{2k-\ell-3} \left(1 + \widehat{U}^{i,N}_{\underline{s}} \right)^2 |\widehat{X}^{i,N}_{\underline{s}}|^{p-4} \\
&\quad + \sum_{m=2}^{2k-\ell-2} \binom{2k-\ell-2}{m} (1 + \frac{1}{\sqrt{N}})^{2k-\ell-2-m} \left(1 + \widehat{U}^{i,N}_{\underline{s}} \right)^m |\widehat{X}^{i,N}_{\underline{s}}|^{p-2-m} \Bigg].
\end{aligned}$$

Set $a = |z|\widetilde{L_3}\left(1+\frac{1}{\sqrt{N}}\right)$. Note that

$$\begin{aligned}&\sum_{\ell=0}^{k}\binom{k}{\ell}2^{\ell}|z|^{2k-\ell}\widetilde{L_3}^{2k-\ell-2}\left(1+\frac{1}{\sqrt{N}}\right)^{2k-\ell-3}\\&=\widetilde{L_3}^{-2}\left(1+\frac{1}{\sqrt{N}}\right)^{-3}\left(a^2+2a\right)^{k}\end{aligned}$$

and

$$\begin{aligned}&-\frac{1}{2}\sum_{\ell=0}^{k}\binom{k}{\ell}2^{\ell}|z|^{2k-\ell}\widetilde{L_3}^{2k-\ell-2}\left(1+\frac{1}{\sqrt{N}}\right)^{2k-\ell-3}\ell\\&=-\widetilde{L_3}^{-2}\left(1+\frac{1}{\sqrt{N}}\right)^{-3}ka\left(a^2+2a\right)^{k-1}.\end{aligned}$$

These facts imply that

$$\begin{aligned}&\sum_{\ell=0}^{k}\binom{k}{\ell}2^{\ell}|z|^{2k-\ell}\widetilde{L_3}^{2k-\ell-2}\left(1+\frac{1}{\sqrt{N}}\right)^{2k-\ell-3}\left(\frac{1}{\sqrt{N}}+k-\frac{\ell}{2}\right)\\&=\widetilde{L_3}^{-2}\left(1+\frac{1}{\sqrt{N}}\right)^{-3}\left(a^2+2a\right)^{k-1}\left(\frac{a^2+2a}{\sqrt{N}}+k(a^2+a)\right).\end{aligned}$$

Moreover, similar to the estimate (3.49), we get

$$\left|c_\Delta(\widehat{X}^{i,N}_{\underline{s}},\mu^{\widehat{\boldsymbol{X}}^N}_{\underline{s}})\right|^2\le 2\widetilde{L_3}^2\left(\left(1+\frac{1}{\sqrt{N}}\right)^2|\widehat{X}^{i,N}_{\underline{s}}|^2+\left(1+\widehat{U}^{i,N}_{\underline{s}}\right)^2\right).$$

Therefore, we have

$$\begin{aligned}&\sum_{\ell=0}^{k}\binom{k}{\ell}2^{\ell}|\widehat{X}^{i,N}_{\underline{s}}|^{p-2k+\ell}\left|c_\Delta(\widehat{X}^{i,N}_{\underline{s}},\mu^{\widehat{\boldsymbol{X}}^N}_{\underline{s}})\right|^{2k-\ell}|z|^{2k-\ell}\\&\le\left|c_\Delta(\widehat{X}^{i,N}_{\underline{s}},\mu^{\widehat{\boldsymbol{X}}^N}_{\underline{s}})\right|^2|\widehat{X}^{i,N}_{\underline{s}}|^{p-2}\widetilde{L_3}^{-2}\left(1+\frac{1}{\sqrt{N}}\right)^{-3}\left(a^2+2a\right)^{k-1}\left(\frac{a^2+2a}{\sqrt{N}}+k(a^2+a)\right)\\&\quad+\sum_{\ell=0}^{k}|z|^{2k-\ell}Q_p\left(p-2,2k-\ell,|\widehat{X}^{i,N}_{\underline{s}}|,1+\widehat{U}^{i,N}_{\underline{s}}\right).\end{aligned}$$

Next, using $\sum_{k=2}^{p/2}\binom{p/2}{k}kz^{k-1}=\frac{p}{2}((1+z)^{p/2-1}-1)$ and $\sum_{k=2}^{p/2}\binom{p/2}{k}z^{k-1}=z^{-1}((1+z)^{p/2}-1-\frac{p}{2}z)$ with $z>0$, we get that

$$\sum_{k=2}^{p/2}\binom{p/2}{k}|\widehat{X}^{i,N}_{\underline{s}}|^{p-2k}\left(\left|c_\Delta(\widehat{X}^{i,N}_{\underline{s}},\mu^{\widehat{\boldsymbol{X}}^N}_{\underline{s}})z\right|^2+2\left\langle\widehat{X}^{i,N}_{\underline{s}},c_\Delta(\widehat{X}^{i,N}_{\underline{s}},\mu^{\widehat{\boldsymbol{X}}^N}_{\underline{s}})z\right\rangle\right)^k$$

$$\leq \left|c_\Delta(\widehat{X}^{i,N}_{\underline{s}},\mu^{\widehat{\boldsymbol{X}}^N}_{\underline{s}})\right|^2 |\widehat{X}^{i,N}_{\underline{s}}|^{p-2}\widetilde{L_3}^{-2}\left(1+\frac{1}{\sqrt{N}}\right)^{-3}$$

$$\times\left(\frac{1}{\sqrt{N}}\Big((1+a)^p-1-\frac{p}{2}(a^2+2a)\Big)+(a^2+a)\frac{p}{2}\Big((1+a)^{p-2}-1\Big)\right)$$

$$+\sum_{k=2}^{p/2}\sum_{\ell=0}^{k}\binom{p/2}{k}|z|^{2k-\ell}Q_p\left(p-2,2k-\ell,|\widehat{X}^{i,N}_{\underline{s}}|,1+\widehat{U}^{i,N}_{\underline{s}}\right).$$

Applying the inequality $(1+a)^p-1-\frac{p}{2}(a^2+2a)\leq(a+1)^2\left((1+a)^{p-2}-1\right)$, with $p\geq 2$ and $a>0$, we get that for $\epsilon=\frac{1}{2\sqrt{N}}\max\{3\widetilde{L_3},1\}$,

$$\left|\widehat{X}^{i,N}_{\underline{s}}+c_\Delta(\widehat{X}^{i,N}_{\underline{s}},\mu^{\widehat{\boldsymbol{X}}^N}_{\underline{s}})z\right|^p-|\widehat{X}^{i,N}_{\underline{s}}|^p-p|\widehat{X}^{i,N}_{\underline{s}}|^{p-2}\left\langle\widehat{X}^{i,N}_{\underline{s}},c_\Delta(\widehat{X}^{i,N}_{\underline{s}},\mu^{\widehat{\boldsymbol{X}}^N}_{\underline{s}})z\right\rangle$$

$$\leq\left|c_\Delta(\widehat{X}^{i,N}_{\underline{s}},\mu^{\widehat{\boldsymbol{X}}^N}_{\underline{s}})\right|^2|\widehat{X}^{i,N}_{\underline{s}}|^{p-2}\Bigg[\frac{p}{2}|z|^2+\widetilde{L_3}^{-2}\left(1+\frac{1}{\sqrt{N}}\right)^{-3}$$

$$\times\left(\frac{1}{\sqrt{N}}\Big((1+a)^p-1-\frac{p}{2}(a^2+2a)\Big)+(a^2+a)\frac{p}{2}\Big((1+a)^{p-2}-1\Big)\right)\Bigg]$$

$$+\sum_{k=2}^{p/2}\sum_{\ell=0}^{k}\binom{p/2}{k}|z|^{2k-\ell}Q_p\left(p-2,2k-\ell,|\widehat{X}^{i,N}_{\underline{s}}|,1+\widehat{U}^{i,N}_{\underline{s}}\right)$$

$$\leq\left|c_\Delta(\widehat{X}^{i,N}_{\underline{s}},\mu^{\widehat{\boldsymbol{X}}^N}_{\underline{s}})\right|^2|\widehat{X}^{i,N}_{\underline{s}}|^{p-2}\Bigg[\frac{p}{2}|z|^2+\widetilde{L_3}^{-2}\left(1+\frac{1}{\sqrt{N}}\right)^{-3}\Big((1+a)^{p-2}-1\Big)(a+1)$$

$$\times\left(\frac{a+1}{\sqrt{N}}+\frac{ap}{2}\right)\Bigg]+\sum_{k=2}^{p/2}\sum_{\ell=0}^{k}\binom{p/2}{k}|z|^{2k-\ell}Q_p\left(p-2,2k-\ell,|\widehat{X}^{i,N}_{\underline{s}}|,1+\widehat{U}^{i,N}_{\underline{s}}\right)$$

$$\leq\left|c_\Delta(\widehat{X}^{i,N}_{\underline{s}},\mu^{\widehat{\boldsymbol{X}}^N}_{\underline{s}})\right|^2|\widehat{X}^{i,N}_{\underline{s}}|^{p-2}p\left[\frac{|z|^2}{2}+\widetilde{L_3}^{-2}\Big((1+a)^{p-1}-a-1\Big)\left(\frac{a+1}{p\sqrt{N}}+\frac{a}{2}\right)\right]$$

$$+\sum_{k=2}^{p/2}\sum_{\ell=0}^{k}\binom{p/2}{k}|z|^{2k-\ell}Q_p\left(p-2,2k-\ell,|\widehat{X}^{i,N}_{\underline{s}}|,1+\widehat{U}^{i,N}_{\underline{s}}\right)$$

$$\leq\left|c_\Delta(\widehat{X}^{i,N}_{\underline{s}},\mu^{\widehat{\boldsymbol{X}}^N}_{\underline{s}})\right|^2|\widehat{X}^{i,N}_{\underline{s}}|^{p-2}p\Bigg[\frac{|z|^2}{2}+\widetilde{L_3}^{-2}\Bigg(\Big(1+|z|(\widetilde{L_3}+\epsilon)\Big)^{p-1}$$

$$-|z|(\widetilde{L_3}+\epsilon)-1\Bigg)\left(|z|\left(\frac{\widetilde{L_3}}{2}+\epsilon\right)+\epsilon\right)\Bigg]+\sum_{k=2}^{p/2}\sum_{\ell=0}^{k}\binom{p/2}{k}|z|^{2k-\ell}$$

$$\times Q_p\left(p-2,2k-\ell,|\widehat{X}^{i,N}_{\underline{s}}|,1+\widehat{U}^{i,N}_{\underline{s}}\right),$$

where we have used the fact that $a\leq|z|(\widetilde{L_3}+\epsilon)$, $\dfrac{a+1}{p\sqrt{N}}+\dfrac{a}{2}\leq|z|\Big(\frac{\widetilde{L_3}}{2}+\epsilon\Big)+\epsilon$. □

Lemma 3.6.8. *Assume Conditions* ***T1***–***T5*** *and* ***C5*** *hold. Then, for any even integer $p \in (0; p_0]$, there exists a positive constant C_p such that for any $s > 0$ and $z \in \mathbb{R}^d$,*

$$\mathbb{E}\left[|\widehat{X}^{i,N}_s|^{p-2}\langle \widehat{X}^{i,N}_s, c_\Delta(\widehat{X}^{i,N}_{\underline{s}}, \mu^{\widehat{\boldsymbol{X}}^N}_{\underline{s}})z\rangle - |\widehat{X}^{i,N}_{\underline{s}}|^{p-2}\langle \widehat{X}^{i,N}_{\underline{s}}, c_\Delta(\widehat{X}^{i,N}_{\underline{s}}, \mu^{\widehat{\boldsymbol{X}}^N}_{\underline{s}})z\rangle \Big| \mathcal{F}_{\underline{s}}\right]$$

$$\leq C_p|z|\left(\sum_{j=0}^{p-2}|\widehat{X}^{i,N}_{\underline{s}}|^j + \mathcal{W}_2(\mu^{\widehat{\boldsymbol{X}}^N}_{\underline{s}}, \delta_0)\sum_{j=0}^{p-3}|\widehat{X}^{i,N}_{\underline{s}}|^j\right). \tag{3.50}$$

Proof. By using the binomial theorem, (3.31) and (3.37) of **T5**, we have

$$\begin{aligned}
&|\widehat{X}^{i,N}_s|^{p-2}\langle \widehat{X}^{i,N}_s, c_\Delta(\widehat{X}^{i,N}_{\underline{s}}, \mu^{\widehat{\boldsymbol{X}}^N}_{\underline{s}})z\rangle - |\widehat{X}^{i,N}_{\underline{s}}|^{p-2}\langle \widehat{X}^{i,N}_{\underline{s}}, c_\Delta(\widehat{X}^{i,N}_{\underline{s}}, \mu^{\widehat{\boldsymbol{X}}^N}_{\underline{s}})z\rangle \\
&= \langle \widehat{X}^{i,N}_{\underline{s}}, c_\Delta(\widehat{X}^{i,N}_{\underline{s}}, \mu^{\widehat{\boldsymbol{X}}^N}_{\underline{s}})z\rangle \left(|\widehat{X}^{i,N}_s|^{p-2} - |\widehat{X}^{i,N}_{\underline{s}}|^{p-2}\right) \\
&\quad + \langle \widehat{X}^{i,N}_s - \widehat{X}^{i,N}_{\underline{s}}, c_\Delta(\widehat{X}^{i,N}_{\underline{s}}, \mu^{\widehat{\boldsymbol{X}}^N}_{\underline{s}})z\rangle |\widehat{X}^{i,N}_s|^{p-2} \\
&= \langle \widehat{X}^{i,N}_{\underline{s}}, c_\Delta(\widehat{X}^{i,N}_{\underline{s}}, \mu^{\widehat{\boldsymbol{X}}^N}_{\underline{s}})z\rangle \sum_{j=1}^{p-2}\binom{p-2}{j}|\widehat{X}^{i,N}_{\underline{s}}|^{p-2-j}\left(|\widehat{X}^{i,N}_s| - |\widehat{X}^{i,N}_{\underline{s}}|\right)^j \\
&\quad + \langle \widehat{X}^{i,N}_s - \widehat{X}^{i,N}_{\underline{s}}, c_\Delta(\widehat{X}^{i,N}_{\underline{s}}, \mu^{\widehat{\boldsymbol{X}}^N}_{\underline{s}})z\rangle |\widehat{X}^{i,N}_{\underline{s}}|^{p-2} \\
&\quad + \langle \widehat{X}^{i,N}_s - \widehat{X}^{i,N}_{\underline{s}}, c_\Delta(\widehat{X}^{i,N}_{\underline{s}}, \mu^{\widehat{\boldsymbol{X}}^N}_{\underline{s}})z\rangle \sum_{j=1}^{p-2}\binom{p-2}{j}|\widehat{X}^{i,N}_{\underline{s}}|^{p-2-j}\left(|\widehat{X}^{i,N}_s| - |\widehat{X}^{i,N}_{\underline{s}}|\right)^j \\
&\leq (p-2)|c_\Delta(\widehat{X}^{i,N}_{\underline{s}}, \mu^{\widehat{\boldsymbol{X}}^N}_{\underline{s}})||z||\widehat{X}^{i,N}_{\underline{s}}|^{p-2}\left|\widehat{X}^{i,N}_s - \widehat{X}^{i,N}_{\underline{s}}\right| \\
&\quad + |c_\Delta(\widehat{X}^{i,N}_{\underline{s}}, \mu^{\widehat{\boldsymbol{X}}^N}_{\underline{s}})||z|\sum_{j=2}^{p-2}\binom{p-2}{j}|\widehat{X}^{i,N}_{\underline{s}}|^{p-1-j}\left|\widehat{X}^{i,N}_s - \widehat{X}^{i,N}_{\underline{s}}\right|^j \\
&\quad + \left\langle b\left(\widehat{X}^{i,N}_{\underline{s}}, \mu^{\widehat{\boldsymbol{X}}^N}_{\underline{s}}\right)(s-\underline{s}), c_\Delta(\widehat{X}^{i,N}_{\underline{s}}, \mu^{\widehat{\boldsymbol{X}}^N}_{\underline{s}})z\right\rangle |\widehat{X}^{i,N}_{\underline{s}}|^{p-2} \\
&\quad + \left\langle \sigma_\Delta\left(\widehat{X}^{i,N}_{\underline{s}}, \mu^{\widehat{\boldsymbol{X}}^N}_{\underline{s}}\right)\left(W^i_s - W^i_{\underline{s}}\right) + c_\Delta\left(\widehat{X}^{i,N}_{\underline{s}}, \mu^{\widehat{\boldsymbol{X}}^N}_{\underline{s}}\right)\left(Z^i_s - Z^i_{\underline{s}}\right), c_\Delta(\widehat{X}^{i,N}_{\underline{s}}, \mu^{\widehat{\boldsymbol{X}}^N}_{\underline{s}})z\right\rangle \\
&\quad \times |\widehat{X}^{i,N}_{\underline{s}}|^{p-2} + |c_\Delta(\widehat{X}^{i,N}_{\underline{s}}, \mu^{\widehat{\boldsymbol{X}}^N}_{\underline{s}})||z|\sum_{j=1}^{p-2}\binom{p-2}{j}|\widehat{X}^{i,N}_{\underline{s}}|^{p-2-j}\left|\widehat{X}^{i,N}_s - \widehat{X}^{i,N}_{\underline{s}}\right|^{j+1} \\
&\leq (p-2)|z||\widehat{X}^{i,N}_{\underline{s}}|^{p-2}\Bigg(|c_\Delta(\widehat{X}^{i,N}_{\underline{s}}, \mu^{\widehat{\boldsymbol{X}}^N}_{\underline{s}})||b(\widehat{X}^{i,N}_{\underline{s}}, \mu^{\widehat{\boldsymbol{X}}^N}_{\underline{s}})|(s-\underline{s}) + |c_\Delta(\widehat{X}^{i,N}_{\underline{s}}, \mu^{\widehat{\boldsymbol{X}}^N}_{\underline{s}})| \\
&\quad \times |\sigma_\Delta\left(\widehat{X}^{i,N}_{\underline{s}}, \mu^{\widehat{\boldsymbol{X}}^N}_{\underline{s}}\right)|\left|W^i_s - W^i_{\underline{s}}\right| + |c_\Delta(\widehat{X}^{i,N}_{\underline{s}}, \mu^{\widehat{\boldsymbol{X}}^N}_{\underline{s}})|^2\left|Z^i_s - Z^i_{\underline{s}}\right|\Bigg) \\
&\quad + \widetilde{L_3}\left(1 + |\widehat{X}^{i,N}_{\underline{s}}| + \mathcal{W}_2(\mu^{\widehat{\boldsymbol{X}}^N}_{\underline{s}}, \delta_0)\right)|z|\sum_{j=2}^{p-2}\binom{p-2}{j}|\widehat{X}^{i,N}_{\underline{s}}|^{p-1-j}3^{j-1}
\end{aligned}$$

$$\times \left[\left| b\left(\widehat{X}_{\underline{s}}^{i,N}, \mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N}\right)\right|^j |s-\underline{s}|^j + \left|\sigma_\Delta\left(\widehat{X}_{\underline{s}}^{i,N}, \mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N}\right)\right|^j |W_s^i - W_{\underline{s}}^i|^j \right.$$

$$\left. + \left|c_\Delta\left(\widehat{X}_{\underline{s}}^{i,N}, \mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N}\right)\right|^j |Z_s^i - Z_{\underline{s}}^i|^j \right] + \left\langle b\left(\widehat{X}_{\underline{s}}^{i,N}, \mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N}\right)(s-\underline{s}), c_\Delta(\widehat{X}_{\underline{s}}^{i,N}, \mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N})z \right\rangle$$

$$\times |\widehat{X}_{\underline{s}}^{i,N}|^{p-2} + \left\langle \sigma_\Delta\left(\widehat{X}_{\underline{s}}^{i,N}, \mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N}\right)\left(W_s^i - W_{\underline{s}}^i\right) + c_\Delta\left(\widehat{X}_{\underline{s}}^{i,N}, \mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N}\right)\right.$$

$$\left. \times \left(Z_s^i - Z_{\underline{s}}^i\right), c_\Delta(\widehat{X}_{\underline{s}}^{i,N}, \mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N})z\right\rangle |\widehat{X}_{\underline{s}}^{i,N}|^{p-2} + \widetilde{L_3}\left(1 + |\widehat{X}_{\underline{s}}^{i,N}| + \mathcal{W}_2(\mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N}, \delta_0)\right)|z|$$

$$\times \sum_{j=1}^{p-2}\binom{p-2}{j}|\widehat{X}_{\underline{s}}^{i,N}|^{p-2-j}3^j\left(\left|b\left(\widehat{X}_{\underline{s}}^{i,N}, \mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N}\right)\right|^{j+1}|s-\underline{s}|^{j+1}\right.$$

$$\left. + \left|\sigma_\Delta\left(\widehat{X}_{\underline{s}}^{i,N}, \mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N}\right)\right|^{j+1}\left|W_s^i - W_{\underline{s}}^i\right|^{j+1} + \left|c_\Delta\left(\widehat{X}_{\underline{s}}^{i,N}, \mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N}\right)\right|^{j+1}\left|Z_s^i - Z_{\underline{s}}^i\right|^{j+1}\right).$$

By applying the estimates (3.45) and condition **T4**, we obtain the desired result. □

Proposition 3.6.9. *Assume Conditions **T1**–**T5** and **C5** hold. Then, for any positive $k \le p_0/2$, there exists a positive constant $C = C(x_0, k, \widetilde{\gamma}_1, \widetilde{\gamma}_2, \widetilde{\eta}, L, \widetilde{L_3}, p_0)$ which depends neither on Δ nor on t such that for any $t \ge 0$,*

$$\max_{i\in\{1,\ldots,N\}}\left(\mathbb{E}\left[|\widehat{X}_t^{i,N}|^{2k}\right] \vee \mathbb{E}\left[|\widehat{X}_{\underline{t}}^{i,N}|^{2k}\right]\right) \le \begin{cases} Ce^{2k\widetilde{\gamma}t} & \text{if } \widetilde{\gamma} > 0, \\ C(1+t)^k & \text{if } \widetilde{\gamma} = 0, \\ C & \text{if } \widetilde{\gamma} < 0, \end{cases} \tag{3.51}$$

where $\widetilde{\gamma} = \widetilde{\gamma}_1 + \widetilde{\gamma}_2$.

Proof. Using Hölder's inequality, it suffices to show (3.51) for positive intergers k with $k \le p_0/2$. We are going to use the induction method. First, note that (3.51) is valid for $k = 1$ due to Lemma 3.6.3.

Next, assume that (3.51) holds for any $k \le k_0 \le [p_0/2] - 1$. We wish to show that (3.51) still holds for $k = k_0 + 1$. For this, using Itô's formula for $e^{-p\lambda t}|\widehat{X}_t^{i,N}|^p$ with even integer $p := 2(k_0+1)$, we have

$$e^{-p\lambda t}\left|\widehat{X}_t^{i,N}\right|^p = |x_0|^p + \int_0^t e^{-p\lambda s}\mathcal{R}_s ds + \mathcal{M}_t, \tag{3.52}$$

where

$$\mathcal{R}_s = -p\lambda|\widehat{X}_s^{i,N}|^p + p|\widehat{X}_s^{i,N}|^{p-2}\left\langle \widehat{X}_s^{i,N}, b(\widehat{X}_{\underline{s}}^{i,N}, \mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N})\right\rangle$$

$$+ \frac{p}{2}|\widehat{X}_s^{i,N}|^{p-2}\left|\sigma_\Delta(\widehat{X}_{\underline{s}}^{i,N}, \mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N})\right|^2 + \frac{p(p-2)}{2}|\widehat{X}_s^{i,N}|^{p-4}\left|(\widehat{X}_s^{i,N})^{\mathsf{T}}\sigma_\Delta(\widehat{X}_{\underline{s}}^{i,N}, \mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N})\right|^2$$

$$+\int_{\mathbb{R}_0^d}\left(\left|\widehat{X}_s^{i,N}+c_\Delta(\widehat{X}_{\underline{s}}^{i,N},\mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N})z\right|^p-|\widehat{X}_s^{i,N}|^p\right.$$
$$\left.-p|\widehat{X}_s^{i,N}|^{p-2}\left\langle\widehat{X}_s^{i,N},c_\Delta(\widehat{X}_{\underline{s}}^{i,N},\mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N})z\right\rangle\right)\nu(dz)$$
$$\mathcal{M}_t=p\int_0^t e^{-p\lambda s}|\widehat{X}_s^{i,N}|^{p-2}\left\langle\widehat{X}_s^{i,N},\sigma_\Delta(\widehat{X}_{\underline{s}}^{i,N},\mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N})dW_s^i\right\rangle$$
$$+\int_0^t\int_{\mathbb{R}_0^d}e^{-p\lambda s}\left(\left|\widehat{X}_{s-}^{i,N}+c_\Delta(\widehat{X}_{\underline{s-}}^{i,N},\mu_{\underline{s-}}^{\widehat{\boldsymbol{X}}^N})z\right|^p-|\widehat{X}_{s-}^{i,N}|^p\right)\widetilde{N}^i(ds,dz).$$

If follows from Lemma 3.6.5 that

$$\mathbb{E}\Bigg[-p\lambda|\widehat{X}_s^{i,N}|^p+p|\widehat{X}_s^{i,N}|^{p-2}\left\langle\widehat{X}_s^{i,N},b(\widehat{X}_{\underline{s}}^{i,N},\mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N})\right\rangle+\frac{p}{2}|\widehat{X}_s^{i,N}|^{p-2}\left|\sigma_\Delta(\widehat{X}_{\underline{s}}^{i,N},\mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N})\right|^2$$
$$+\frac{p(p-2)}{2}|\widehat{X}_s^{i,N}|^{p-4}\left|(\widehat{X}_s^{i,N})^\mathsf{T}\sigma_\Delta(\widehat{X}_{\underline{s}}^{i,N},\mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N})\right|^2\Big|\mathcal{F}_{\underline{s}}\Bigg]$$
$$\leq p|\widehat{X}_{\underline{s}}^{i,N}|^{p-2}\left(-\lambda|\widehat{X}_{\underline{s}}^{i,N}|^2+\left\langle\widehat{X}_{\underline{s}}^{i,N},b(\widehat{X}_{\underline{s}}^{i,N},\mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N})\right\rangle+\frac{p-1}{2}|\sigma_\Delta(\widehat{X}_{\underline{s}}^{i,N},\mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N})|^2\right)$$
$$+\overline{Q}_{p-2}\left(\left|\widehat{X}_{\underline{s}}^{i,N}\right|\right),\tag{3.53}$$

where $\overline{Q}_{p-2}(|\widehat{X}_{\underline{s}}^{i,N}|)$ is a polynomial in $|\widehat{X}_{\underline{s}}^{i,N}|$ of degree $p-2$.

We write

$$\left|\widehat{X}_s^{i,N}+c_\Delta(\widehat{X}_{\underline{s}}^{i,N},\mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N})z\right|^p-|\widehat{X}_s^{i,N}|^p-p|\widehat{X}_s^{i,N}|^{p-2}\left\langle\widehat{X}_s^{i,N},c_\Delta(\widehat{X}_{\underline{s}}^{i,N},\mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N})z\right\rangle$$
$$=\left|\widehat{X}_{\underline{s}}^{i,N}+c_\Delta(\widehat{X}_{\underline{s}}^{i,N},\mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N})z\right|^p-|\widehat{X}_{\underline{s}}^{i,N}|^p-p|\widehat{X}_{\underline{s}}^{i,N}|^{p-2}\left\langle\widehat{X}_{\underline{s}}^{i,N},c_\Delta(\widehat{X}_{\underline{s}}^{i,N},\mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N})z\right\rangle$$
$$+\left(\left|\widehat{X}_s^{i,N}+c_\Delta(\widehat{X}_{\underline{s}}^{i,N},\mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N})z\right|^p-\left|\widehat{X}_{\underline{s}}^{i,N}+c_\Delta(\widehat{X}_{\underline{s}}^{i,N},\mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N})z\right|^p\right)-\left(|\widehat{X}_s^{i,N}|^p-|\widehat{X}_{\underline{s}}^{i,N}|^p\right)$$
$$-p\left(|\widehat{X}_s^{i,N}|^{p-2}\left\langle\widehat{X}_s^{i,N},c_\Delta(\widehat{X}_{\underline{s}}^{i,N},\mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N})z\right\rangle-|\widehat{X}_{\underline{s}}^{i,N}|^{p-2}\left\langle\widehat{X}_{\underline{s}}^{i,N},c_\Delta(\widehat{X}_{\underline{s}}^{i,N},\mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N})z\right\rangle\right).\tag{3.54}$$

Therefore, taking the conditional expectation on both sides of (3.54) and inserting (3.47), (3.48), and (3.50) into the right hand side, we obtain that for $\epsilon=\frac{1}{2\sqrt{N}}\max\{3\widetilde{L_3},1\}$,

$$\mathbb{E}\left[\left|\widehat{X}_s^{i,N}+c_\Delta\left(\widehat{X}_{\underline{s}}^{i,N},\mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N}\right)z\right|^p-|\widehat{X}_s^{i,N}|^p-p|\widehat{X}_s^{i,N}|^{p-2}\left\langle\widehat{X}_s^{i,N},c_\Delta\left(\widehat{X}_{\underline{s}}^{i,N},\mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N}\right)z\right\rangle\Big|\mathcal{F}_{\underline{s}}\right]$$
$$\leq p\left|c_\Delta(\widehat{X}_{\underline{s}}^{i,N},\mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N})\right|^2|\widehat{X}_{\underline{s}}^{i,N}|^{p-2}\Bigg[\frac{|z|^2}{2}+\widetilde{L_3}^{-2}\left(\left(1+|z|(\widetilde{L_3}+\epsilon)\right)^{p-1}-|z|(\widetilde{L_3}+\epsilon)-1\right)$$
$$\times\left(|z|\left(\frac{\widetilde{L_3}}{2}+\epsilon\right)+\epsilon\right)\Bigg]+\sum_{k=2}^{p/2}\sum_{\ell=0}^{k}\binom{p/2}{k}|z|^{2k-\ell}Q_p\left(p-2,2k-\ell,|\widehat{X}_{\underline{s}}^{i,N}|,1+\widehat{U}_{\underline{s}}^{i,N}\right)$$

$$+\widehat{Q}_{p-2}\left(\left|\widehat{X}_{\underline{s}}^{i,N}\right|,|z|\right)+\sum_{j=1}^{p/2}\sum_{k=0}^{j}C_{j,k}|z|^{j+k}\mathcal{W}_2^{j+k}(\mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N},\delta_0)\Bigg[\left|\widehat{X}_{\underline{s}}^{i,N}\right|^{j-k-2}$$
$$+\left|\widehat{X}_{\underline{s}}^{i,N}\right|^{p-j-k-2}+\sum_{\ell=2}^{j-k}\left(\left|\widehat{X}_{\underline{s}}^{i,N}\right|^{j-k-\ell}+\left|\widehat{X}_{\underline{s}}^{i,N}\right|^{p-j-k-\ell}\right)+\sum_{\ell=2}^{p-2j}\left|\widehat{X}_{\underline{s}}^{i,N}\right|^{p-j-k-\ell}\Bigg]$$
$$+C_p|z|\left(\sum_{j=0}^{p-2}|\widehat{X}_{\underline{s}}^{i,N}|^j+\mathcal{W}_2(\mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N},\delta_0)\sum_{j=0}^{p-3}|\widehat{X}_{\underline{s}}^{i,N}|^j\right). \tag{3.55}$$

Hence, combining (3.53) and (3.55), we get

$$\mathbb{E}[\mathcal{R}_s|\mathcal{F}_{\underline{s}}]$$
$$\leq p|\widehat{X}_{\underline{s}}^{i,N}|^{p-2}\Bigg[-\lambda|\widehat{X}_{\underline{s}}^{i,N}|^2+\langle\widehat{X}_{\underline{s}}^{i,N},b(\widehat{X}_{\underline{s}}^{i,N},\mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N})\rangle+\frac{p-1}{2}|\sigma_\Delta(\widehat{X}_{\underline{s}}^{i,N},\mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N})|^2$$
$$+\left|c_\Delta\left(\widehat{X}_{\underline{s}}^{i,N},\mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N}\right)\right|^2\int_{\mathbb{R}_0^d}\left(\frac{|z|^2}{2}+\widetilde{L_3}^{-2}\left(\left(1+|z|(\widetilde{L_3}+\epsilon)\right)^{p-1}-|z|(\widetilde{L_3}+\epsilon)-1\right)\right.$$
$$\left.\times\left(|z|\Big(\frac{\widetilde{L_3}}{2}+\epsilon\Big)+\epsilon\right)\right)\nu(dz)\Bigg]+\overline{Q}_{p-2}\left(\left|\widehat{X}_{\underline{s}}^{i,N}\right|\right)+\int_{\mathbb{R}_0^d}\Bigg[\sum_{k=2}^{p/2}\sum_{\ell=0}^{k}\binom{p/2}{k}|z|^{2k-\ell}$$
$$\times Q_p\left(p-2,2k-\ell,|\widehat{X}_{\underline{s}}^{i,N}|,1+\widehat{U}_{\underline{s}}^{i,N}\right)+\widehat{Q}_{p-2}\left(\left|\widehat{X}_{\underline{s}}^{i,N}\right|,|z|\right)$$
$$+\sum_{j=1}^{p/2}\sum_{k=0}^{j}C_{j,k}|z|^{j+k}\mathcal{W}_2^{j+k}(\mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N},\delta_0)\Bigg(\left|\widehat{X}_{\underline{s}}^{i,N}\right|^{j-k-2}+\left|\widehat{X}_{\underline{s}}^{i,N}\right|^{p-j-k-2}$$
$$+\sum_{\ell=2}^{j-k}\left(\left|\widehat{X}_{\underline{s}}^{i,N}\right|^{j-k-\ell}+\left|\widehat{X}_{\underline{s}}^{i,N}\right|^{p-j-k-\ell}\right)+\sum_{\ell=2}^{p-2j}\left|\widehat{X}_{\underline{s}}^{i,N}\right|^{p-j-k-\ell}\Bigg)$$
$$+C_p|z|\left(\sum_{j=0}^{p-2}|\widehat{X}_{\underline{s}}^{i,N}|^j+\mathcal{W}_2(\mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N},\delta_0)\sum_{j=0}^{p-3}|\widehat{X}_{\underline{s}}^{i,N}|^j\right)\Bigg]\nu(dz).$$

In the following, we choose $\lambda=\widetilde{\gamma}_1+\frac{\widetilde{\gamma}_2}{N}$. It follows from (3.38) of **T5** and the equality $\mathcal{W}_2^2(\mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N},\delta_0)=\frac{1}{N}\sum_{m=1}^N|\widehat{X}_{\underline{s}}^{m,N}|^2$ that

$$-(\widetilde{\gamma}_1+\frac{\widetilde{\gamma}_2}{N})|\widehat{X}_{\underline{s}}^{i,N}|^2+\langle\widehat{X}_{\underline{s}}^{i,N},b(\widehat{X}_{\underline{s}}^{i,N},\mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N})\rangle+\frac{p-1}{2}|\sigma_\Delta(\widehat{X}_{\underline{s}}^{i,N},\mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N})|^2$$
$$+\left|c_\Delta\left(\widehat{X}_{\underline{s}}^{i,N},\mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N}\right)\right|^2\int_{\mathbb{R}_0^d}\Bigg[\frac{|z|^2}{2}+\widetilde{L_3}^{-2}\left(\left(1+|z|(\widetilde{L_3}+\epsilon)\right)^{p-1}-|z|(\widetilde{L_3}+\epsilon)-1\right)$$
$$\times\left(|z|\Big(\frac{\widetilde{L_3}}{2}+\epsilon\Big)+\epsilon\right)\Bigg]\nu(dz)$$

$$\leq -(\widetilde{\gamma}_1+\frac{\widetilde{\gamma}_2}{N})|\widehat{X}^{i,N}_{\underline{s}}|^2+\widetilde{\gamma}_1|\widehat{X}^{i,N}_{\underline{s}}|^2+\widetilde{\gamma}_2\mathcal{W}_2^2(\mu^{\widehat{\boldsymbol{X}}^N}_{\underline{s}},\delta_0)+\widetilde{\eta}=\frac{\widetilde{\gamma}_2}{N}\sum_{m=1,m\neq i}^{N}|\widehat{X}^{m,N}_{\underline{s}}|^2+\widetilde{\eta}.$$

Therefore, using the estimate $\mathcal{W}_2(\mu^{\widehat{\boldsymbol{X}}^N}_{\underline{s}},\delta_0)\leq\frac{1}{\sqrt{N}}\sum_{m=1}^N|\widehat{X}^{m,N}_{\underline{s}}|$, we obtain that

$$\begin{aligned}\mathbb{E}[\mathcal{R}_s]\leq p\mathbb{E}\Bigg[&\frac{\widetilde{\gamma}_2}{N}|\widehat{X}^{i,N}_{\underline{s}}|^{p-2}\sum_{m=1,m\neq i}^{N}|\widehat{X}^{m,N}_{\underline{s}}|^2+\int_{\mathbb{R}^d_0}\sum_{k=2}^{p/2}\sum_{\ell=0}^{k}\binom{p/2}{k}|z|^{2k-\ell}\\&\times Q_p\left(p-2,2k-\ell,|\widehat{X}^{i,N}_{\underline{s}}|,1+\widehat{U}^{i,N}_{\underline{s}}\right)\nu(dz)\Bigg]+C\sum_{j=0}^{p-2}\mathbb{E}\left[|\widehat{X}^{i,N}_{\underline{s}}|^j\right],\end{aligned}\tag{3.56}$$

for some positive constant C.

Thanks to Lemma 3.6.1, $\mathbb{E}[\mathcal{M}_t]=0$. Now, we are going to take the expectation for (3.52) with $\lambda=\widetilde{\gamma}_1+\frac{\widetilde{\gamma}_2}{N}$, plug (3.56) into (3.52), and use the inductive assumption, Condition **C5** and the following fact thanks to the independence between $|\widehat{X}^{i,N}_{\underline{s}}|^{p-2}$ and $\sum_{m=1,m\neq i}^N|\widehat{X}^{m,N}_{\underline{s}}|^2$

$$\begin{aligned}&\mathbb{E}\left[|\widehat{X}^{i,N}_{\underline{s}}|^{p-2}\sum_{m=1,m\neq i}^{N}|\widehat{X}^{m,N}_{\underline{s}}|^2\right]=\sum_{m=1,m\neq i}^{N}\mathbb{E}\left[|\widehat{X}^{i,N}_{\underline{s}}|^{p-2}\right]\mathbb{E}\left[|\widehat{X}^{m,N}_{\underline{s}}|^2\right],\\&\mathbb{E}\left[|\widehat{X}^{i,N}_{\underline{s}}|^{p-\ell}\left(1+\widehat{U}^{i,N}_{\underline{s}}\right)^{\ell}\right]=\mathbb{E}\left[|\widehat{X}^{i,N}_{\underline{s}}|^{p-\ell}\right]\mathbb{E}\left[\left(1+\widehat{U}^{i,N}_{\underline{s}}\right)^{\ell}\right],\end{aligned}\tag{3.57}$$

for any $\ell\in\{2,\dots,p-2\}$, where recall that $\widehat{U}^{i,N}_{\underline{s}}=\frac{1}{\sqrt{N}}\sum_{m=1;m\neq i}^N|\widehat{X}^{m,N}_{\underline{s}}|$. As a consequence, we get that

$$\begin{aligned}&\mathbb{E}\left[e^{-p\left(\widetilde{\gamma}_1+\frac{\widetilde{\gamma}_2}{N}\right)t}\left|\widehat{X}^{i,N}_t\right|^p\right]\\&\leq|x_0|^p+C\int_0^t e^{-p\left(\widetilde{\gamma}_1+\frac{\widetilde{\gamma}_2}{N}\right)s}\left(\sum_{\ell=2}^{p-2}\mathbb{E}\left[|\widehat{X}^{i,N}_{\underline{s}}|^{p-\ell}\right]\mathbb{E}\left[\left(1+\widehat{U}^{i,N}_{\underline{s}}\right)^{\ell}\right]+\sum_{j=0}^{p-2}\mathbb{E}\left[|\widehat{X}^{i,N}_{\underline{s}}|^j\right]\right)ds.\end{aligned}$$

Here, recall that $\widetilde{\gamma}=\widetilde{\gamma}_1+\widetilde{\gamma}_2$.

<u>Case $\widetilde{\gamma}>0$:</u>

$$\begin{aligned}&\mathbb{E}\left[e^{-p\left(\widetilde{\gamma}_1+\frac{\widetilde{\gamma}_2}{N}\right)t}\left|\widehat{X}^{i,N}_t\right|^p\right]\\&\leq|x_0|^p+C\int_0^t e^{-p\left(\widetilde{\gamma}_1+\frac{\widetilde{\gamma}_2}{N}\right)s}\left(\sum_{\ell=2}^{p-2}e^{(p-\ell)\left(\widetilde{\gamma}_1+\widetilde{\gamma}_2\right)s}e^{\ell\left(\widetilde{\gamma}_1+\widetilde{\gamma}_2\right)s}+\sum_{j=0}^{p-2}e^{j\left(\widetilde{\gamma}_1+\widetilde{\gamma}_2\right)s}\right)ds\\&\leq|x_0|^p+C\int_0^t e^{-p\left(\widetilde{\gamma}_1+\frac{\widetilde{\gamma}_2}{N}\right)s}e^{p\left(\widetilde{\gamma}_1+\widetilde{\gamma}_2\right)s}ds\end{aligned}$$

$$= |x_0|^p + C\int_0^t e^{-p\widetilde{\gamma}_2\left(\frac{1}{N}-1\right)s}ds$$

$$= |x_0|^p + \frac{C}{-p\widetilde{\gamma}_2\left(\frac{1}{N}-1\right)}\left(e^{-p\widetilde{\gamma}_2\left(\frac{1}{N}-1\right)t}-1\right),$$

which implies that

$$\mathbb{E}\left[\left|\widehat{X}_t^{i,N}\right|^p\right] \le |x_0|^p e^{p\left(\widetilde{\gamma}_1+\frac{\widetilde{\gamma}_2}{N}\right)t} + \frac{C}{p\widetilde{\gamma}_2\left(1-\frac{1}{N}\right)}\left(e^{p\left(\widetilde{\gamma}_1+\widetilde{\gamma}_2\right)t} - e^{p\left(\widetilde{\gamma}_1+\frac{\widetilde{\gamma}_2}{N}\right)t}\right)$$

$$\le Ce^{p\left(\widetilde{\gamma}_1+\widetilde{\gamma}_2\right)t}$$

$$= Ce^{p\widetilde{\gamma}t}.$$

Case $\widetilde{\gamma}=0$: In this case, we have $\widetilde{\gamma}_1+\frac{\widetilde{\gamma}_2}{N}<\widetilde{\gamma}=0$. Using the integration by parts formula repeatedly, it can be checked that for any $\beta<0$ and $q\in\mathbb{N}^*$,

$$\int_0^t e^{-\beta s}(1+s)^q ds \le C(\beta,q)\left(e^{-\beta t}(1+t)^q + \int_0^t e^{-\beta s}ds\right)$$

$$\le C(\beta,q)\left(e^{-\beta t}(1+t)^q + e^{-\beta t}\right)$$

$$\le C(\beta,q)e^{-\beta t}(1+t)^q,$$

for some constant $C(\beta,q)>0$. This deduces that

$$\mathbb{E}\left[e^{-p\left(\widetilde{\gamma}_1+\frac{\widetilde{\gamma}_2}{N}\right)t}\left|\widehat{X}_t^{i,N}\right|^p\right]$$

$$\le |x_0|^p + C\int_0^t e^{-p\left(\widetilde{\gamma}_1+\frac{\widetilde{\gamma}_2}{N}\right)s}\left(\sum_{\ell=2}^{p-2}(1+s)^{(p-\ell)/2}(1+s)^{\ell/2} + \sum_{j=0}^{p-2}(1+s)^{j/2}\right)ds$$

$$\le |x_0|^p + C\int_0^t e^{-p\left(\widetilde{\gamma}_1+\frac{\widetilde{\gamma}_2}{N}\right)s}(1+s)^{p/2}ds$$

$$\le |x_0|^p + Ce^{-p\left(\widetilde{\gamma}_1+\frac{\widetilde{\gamma}_2}{N}\right)t}(1+t)^{p/2}.$$

Therefore, we obtain that

$$\mathbb{E}\left[\left|\widehat{X}_t^{i,N}\right|^p\right] \le C\left(1+t\right)^{p/2}.$$

Case $\widetilde{\gamma}<0$: In this case, we have $\widetilde{\gamma}_1+\frac{\widetilde{\gamma}_2}{N}<\widetilde{\gamma}<0$. Thus, we get

$$\mathbb{E}\left[e^{-p\left(\widetilde{\gamma}_1+\frac{\widetilde{\gamma}_2}{N}\right)t}\left|\widehat{X}_t^{i,N}\right|^p\right] \le |x_0|^p + C\int_0^t e^{-p\left(\widetilde{\gamma}_1+\frac{\widetilde{\gamma}_2}{N}\right)s}ds,$$

which implies that

$$\mathbb{E}\left[\left|\widehat{X}_t^{i,N}\right|^p\right] \le C.$$

Consequently, combining with (3.43), we conclude that (3.51) holds for $k = k_0 + 1$. Thus, the result follows. □

Remark 3.6.10. When condition **C6** and all conditions of Proposition 3.6.9 are satisfied, we get the following estimate on the expectation of the number of timesteps N_T

$$\mathbb{E}\left[N_T - 1\right] \le \frac{C}{\Delta}, \tag{3.58}$$

for any $T > 0$, where the positive constant C does not depend on Δ.

Indeed, the same argument in the proof of Lemma 2 in [18] yields that

$$N_T = \sum_{k=1}^{N_T} 1 \le \int_0^T \frac{1}{\Delta \mathbf{h}(\widehat{\boldsymbol{X}}_{\underline{t}}^N, \mu_{\underline{t}}^{\widehat{\boldsymbol{X}}^N})} dt + 1.$$

Then, using (3.22), Assumption **C6**, and Remark 3.2.1 (i) and (iii), we get that for any $i \in \{1, \dots, N\}$ and $p_0 \ge 2(\ell + 1)$,

$$\begin{aligned}\frac{1}{h(\widehat{X}_{\underline{t}}^{i,N}, \mu_{\underline{t}}^{\widehat{\boldsymbol{X}}^N})} &\le C\left(1 + |\widehat{X}_{\underline{t}}^{i,N}|^{p_0} + \mathcal{W}_2^{p_0}(\mu_{\underline{t}}^{\widehat{\boldsymbol{X}}^N}, \delta_0)\right) \\ &\le C\left(1 + |\widehat{X}_{\underline{t}}^{i,N}|^{p_0} + \frac{1}{N}\sum_{m=1}^{N}\left|\widehat{X}_{\underline{t}}^{m,N}\right|^{p_0}\right).\end{aligned}$$

This, combined with $\mathbf{h}(\widehat{\boldsymbol{X}}_{\underline{t}}^N, \mu_{\underline{t}}^{\widehat{\boldsymbol{X}}^N}) = \min\{h(\widehat{X}_{\underline{t}}^{1,N}, \mu_{\underline{t}}^{\widehat{\boldsymbol{X}}^N}), \dots, h(\widehat{X}_{\underline{t}}^{N,N}, \mu_{\underline{t}}^{\widehat{\boldsymbol{X}}^N})\}$, Lemma 3.6.1 and Proposition 3.6.9, shows the estimate (3.58).

3.7 Convergence

In this section, we will assess the strong convergence rate of the tamed-adaptive Euler-Maruyama scheme, across both finite and infinite time intervals. To accomplish this, we first require the following uniformly-in-time bound for the difference between the two approximate solutions $\widehat{X}_t^{i,N}$ and $\widehat{X}_{\underline{t}}^{i,N}$.

Lemma 3.7.1. *Let all conditions of Proposition 3.6.9 be satisfied. Then for any $p \in [2, p_0]$, there exists a positive constant $C_p = C(p, L)$ such that*

$$\max_{i \in \{1,\dots,N\}} \mathbb{E}\left[\left|\widehat{X}_t^{i,N} - \widehat{X}_{\underline{t}}^{i,N}\right|^p \Big| \mathcal{F}_{\underline{t}}\right] \le C_p \Delta,$$

for any $t \ge 0$ and

$$\max_{i \in \{1,\dots,N\}} \sup_{t \ge 0} \mathbb{E}\left[\left|\widehat{X}_t^{i,N} - \widehat{X}_{\underline{t}}^{i,N}\right|^p\right] \le C_p \Delta.$$

Proof. From (3.31), we have that for any $p \geq 2$,

$$
\begin{aligned}
&|\widehat{X}_t^{i,N} - \widehat{X}_{\underline{t}}^{i,N}|^p \\
&= \left| b(\widehat{X}_{\underline{t}}^{i,N}, \mu_{\underline{t}}^{\widehat{\boldsymbol{X}}^N})(t-\underline{t}) + \sigma_\Delta(\widehat{X}_{\underline{t}}^{i,N}, \mu_{\underline{t}}^{\widehat{\boldsymbol{X}}^N})(W_t^i - W_{\underline{t}}^i) + c_\Delta(\widehat{X}_{\underline{t}}^{i,N}, \mu_{\underline{t}}^{\widehat{\boldsymbol{X}}^N})(Z_t^i - Z_{\underline{t}}^i) \right|^p \\
&\leq 3^{p-1} \left[\left| b(\widehat{X}_{\underline{t}}^{i,N}, \mu_{\underline{t}}^{\widehat{\boldsymbol{X}}^N}) \right|^p (t-\underline{t})^p + \left| \sigma_\Delta(\widehat{X}_{\underline{t}}^{i,N}, \mu_{\underline{t}}^{\widehat{\boldsymbol{X}}^N}) \right|^p \left| W_t^i - W_{\underline{t}}^i \right|^p \right. \\
&\quad \left. + \left| c_\Delta(\widehat{X}_{\underline{t}}^{i,N}, \mu_{\underline{t}}^{\widehat{\boldsymbol{X}}^N}) \right|^p \left| Z_t^i - Z_{\underline{t}}^i \right|^p \right].
\end{aligned}
$$

This, combined with (3.45), concludes the desired result. □

Next, the following additional condition will be needed.

T6. There exists a positive constant L_4 such that

$$
\begin{aligned}
&|\sigma(x,\mu) - \sigma_\Delta(x,\mu)| \leq L_4 \Delta^{1/2} |\sigma(x,\mu)|^2 (1+|x|), \\
&|c(x,\mu) - c_\Delta(x,\mu)| \leq L_4 \Delta^{1/2} |c(x,\mu)|^2 \left(1 + |x| + |b(x,\mu)|\right),
\end{aligned}
$$

for all $x \in \mathbb{R}^d$ and $\mu \in \mathcal{P}_2(\mathbb{R}^d)$.

Remark 3.7.2. If we choose

$$
\sigma_\Delta(x,\mu) = \frac{\sigma(x,\mu)}{1 + \Delta^{1/2}|\sigma(x,\mu)|(1+|x|)}, \tag{3.59}
$$

$$
c_\Delta(x,\mu) = \frac{c(x,\mu)}{1 + \Delta^{1/2}|c(x,\mu)|(1+|x|+|b(x,\mu)|)}, \tag{3.60}
$$

then Conditions **T3, T4** and **T6** are satisfied.

Due to the two-step approximation process necessitated by McKean-Vlasov stochastic differential equations, assessing the convergence rate of the approximation scheme requires us to evaluate both the error between the exact solution X_t and the approximate solution $X_t^{i,N}$ of Propagation of Chaos, as presented in Section 3.4, and the error between the two approximate solutions $X_t^{i,N}$ and $\widehat{X}_t^{i,N}$, which we will address subsequently.

Theorem 3.7.3. *Assume that the coefficients $b, \sigma, c, \sigma_\Delta, c_\Delta$ and the Lévy measure ν satisfy Conditions **C1**, **C3–C5**, **T2–T6**, and $p_0 \geq 4\ell + 6$, $N \geq \left(\frac{\max\{3\widetilde{L}_3, 1\}}{2\epsilon}\right)^2$. Assume further that there exists a constant $\varepsilon > 0$ such that **C2** holds for $\kappa_1 = \kappa_2 = 1 + \varepsilon$, $L_1 \in \mathbb{R}$, $L_2 \geq 0$. Then for any $T > 0$, there exist positive constants $C_T =$*

$C(x_0, L, L_1, L_2, L_4, \widetilde{\gamma}_1, \widetilde{\gamma}_2, \widetilde{\eta}, \widetilde{L_3}, \varepsilon, T)$ *and* $C'_T = C'(x_0, L, L_1, L_2, L_4, \widetilde{\gamma}_1, \widetilde{\gamma}_2, \widetilde{\eta}, \widetilde{L_3}, \varepsilon, T)$ *such that*

$$\max_{i\in\{1,\ldots,N\}} \sup_{t\in[0,T]} \mathbb{E}\left[\left|X_t^{i,N} - \widehat{X}_t^{i,N}\right|^2\right] \le C_T\Delta, \tag{3.61}$$

and for any $p \in (0,2)$,

$$\max_{i\in\{1,\ldots,N\}} \mathbb{E}\left[\sup_{t\in[0,T]} \left|X_t^{i,N} - \widehat{X}_t^{i,N}\right|^p\right] \le \left(\frac{4-p}{2-p}\right)(C'_T\Delta)^{p/2}. \tag{3.62}$$

Moreover, if $\widetilde{\gamma} = \widetilde{\gamma}_1 + \widetilde{\gamma}_2 < 0$*, and* $L_1 + L_2 < 0$*, then, there exists a positive constant* $C'' = C''(x_0, L, L_1, L_2, L_4, \widetilde{\gamma}_1, \widetilde{\gamma}_2, \widetilde{\eta}, \widetilde{L_3}, \varepsilon)$*, which does not depend on* T*, such that*

$$\max_{i\in\{1,\ldots,N\}} \sup_{t\ge 0} \mathbb{E}\left[\left|X_t^{i,N} - \widehat{X}_t^{i,N}\right|^2\right] \le C''\Delta. \tag{3.63}$$

Proof. Thanks to (3.16) and (3.32), we have that for any $i \in \{1,\ldots,N\}$,

$$\begin{aligned} X_t^{i,N} - \widehat{X}_t^{i,N} = &\int_0^t \left(b(X_s^{i,N}, \mu_s^{\boldsymbol{X}^N}) - b\left(\widehat{X}_{\underline{s}}^{i,N}, \mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N}\right)\right) ds \\ &+ \int_0^t \left(\sigma(X_s^{i,N}, \mu_s^{\boldsymbol{X}^N}) - \sigma_\Delta\left(\widehat{X}_{\underline{s}}^{i,N}, \mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N}\right)\right) dW_s^i \\ &+ \int_0^t \int_{\mathbb{R}_0^d} \left(c(X_{s-}^{i,N}, \mu_{s-}^{\boldsymbol{X}^N}) - c_\Delta\left(\widehat{X}_{\underline{s}}^{i,N}, \mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N}\right)\right) z\widetilde{N}^i(ds,dz). \end{aligned}$$

Thanks to Itô's formula, we obtain that for any $\lambda \in \mathbb{R}$,

$$\begin{aligned} &e^{-\lambda t}|X_t^{i,N} - \widehat{X}_t^{i,N}|^2 \\ &= \int_0^t e^{-\lambda s}\Bigg[-\lambda|X_s^{i,N} - \widehat{X}_s^{i,N}|^2 + 2\left\langle X_s^{i,N} - \widehat{X}_s^{i,N}, b(X_s^{i,N}, \mu_s^{\boldsymbol{X}^N}) - b\left(\widehat{X}_{\underline{s}}^{i,N}, \mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N}\right)\right\rangle \\ &+ \left|\sigma(X_s^{i,N}, \mu_s^{\boldsymbol{X}^N}) - \sigma_\Delta\left(\widehat{X}_{\underline{s}}^{i,N}, \mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N}\right)\right|^2\Bigg] ds \\ &+ 2\int_0^t e^{-\lambda s}\left\langle X_s^{i,N} - \widehat{X}_s^{i,N}, \left(\sigma(X_s^{i,N}, \mu_s^{\boldsymbol{X}^N}) - \sigma_\Delta(\widehat{X}_{\underline{s}}^{i,N}, \mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N})\right)dW_s^i\right\rangle \\ &+ \int_0^t \int_{\mathbb{R}_0^d} e^{-\lambda s}\left[|X_{s-}^{i,N} - \widehat{X}_{s-}^{i,N} + \left(c(X_{s-}^{i,N}, \mu_{s-}^{\boldsymbol{X}^N}) - c_\Delta(\widehat{X}_{\underline{s-}}^{i,N}, \mu_{\underline{s-}}^{\widehat{\boldsymbol{X}}^N})\right)z|^2 - |X_{s-}^{i,N} - \widehat{X}_{s-}^{i,N}|^2\right] \\ &\times \widetilde{N}^i(ds,dz) + \int_0^t \int_{\mathbb{R}_0^d} e^{-\lambda s}\left|\left(c(X_s^{i,N}, \mu_s^{\boldsymbol{X}^N}) - c_\Delta(\widehat{X}_{\underline{s}}^{i,N}, \mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N})\right)z\right|^2 \nu(dz)ds. \end{aligned} \tag{3.64}$$

In the following, we will give upper bounds for each term on the right hand side of (3.64). First, we decompose

$$2\left\langle X_s^{i,N} - \widehat{X}_s^{i,N}, b(X_s^{i,N}, \mu_s^{\boldsymbol{X}^N}) - b\left(\widehat{X}_{\underline{s}}^{i,N}, \mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N}\right)\right\rangle$$

$$= 2\left\langle X_s^{i,N} - \widehat{X}_s^{i,N}, b(X_s^{i,N}, \mu_s^{\boldsymbol{X}^N}) - b\left(\widehat{X}_s^{i,N}, \mu_s^{\widehat{\boldsymbol{X}}^N}\right)\right\rangle + S, \tag{3.65}$$

where $S = 2\left\langle X_s^{i,N} - \widehat{X}_s^{i,N}, b\left(\widehat{X}_s^{i,N}, \mu_s^{\widehat{\boldsymbol{X}}^N}\right) - b\left(\widehat{X}_{\underline{s}}^{i,N}, \mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N}\right)\right\rangle$. By using Cauchy's inequality and Condition **C4**, we obtain that for any $\varepsilon_1 > 0$,

$$\begin{aligned}
S &\leq \varepsilon_1 \left|X_s^{i,N} - \widehat{X}_s^{i,N}\right|^2 + \frac{1}{\varepsilon_1}\left|b\left(\widehat{X}_s^{i,N}, \mu_s^{\widehat{\boldsymbol{X}}^N}\right) - b\left(\widehat{X}_{\underline{s}}^{i,N}, \mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N}\right)\right|^2 \\
&\leq \varepsilon_1 \left|X_s^{i,N} - \widehat{X}_s^{i,N}\right|^2 + \frac{6}{\varepsilon_1}L^2\left(1 + \left|\widehat{X}_s^{i,N}\right|^{2\ell} + \left|\widehat{X}_{\underline{s}}^{i,N}\right|^{2\ell}\right) \\
&\quad \times \left(\left|\widehat{X}_s^{i,N} - \widehat{X}_{\underline{s}}^{i,N}\right|^2 + \mathcal{W}_2^2\left(\mu_s^{\widehat{\boldsymbol{X}}^N}, \mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N}\right)\right) \\
&\leq \varepsilon_1 \left|X_s^{i,N} - \widehat{X}_s^{i,N}\right|^2 + \frac{6}{\varepsilon_1}L^2\left(1 + 2^{2\ell-1}\left(\left|\widehat{X}_s^{i,N} - \widehat{X}_{\underline{s}}^{i,N}\right|^{2\ell} + \left|\widehat{X}_{\underline{s}}^{i,N}\right|^{2\ell}\right) + \left|\widehat{X}_{\underline{s}}^{i,N}\right|^{2\ell}\right) \\
&\quad \times \left(\left|\widehat{X}_s^{i,N} - \widehat{X}_{\underline{s}}^{i,N}\right|^2 + \mathcal{W}_2^2\left(\mu_s^{\widehat{\boldsymbol{X}}^N}, \mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N}\right)\right) \\
&= \varepsilon_1 \left|X_s^{i,N} - \widehat{X}_s^{i,N}\right|^2 + \frac{6}{\varepsilon_1}L^2\left[\left|\widehat{X}_s^{i,N} - \widehat{X}_{\underline{s}}^{i,N}\right|^2 + \mathcal{W}_2^2\left(\mu_s^{\widehat{\boldsymbol{X}}^N}, \mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N}\right)\right. \\
&\quad \left. + 2^{2\ell-1}\left|\widehat{X}_s^{i,N} - \widehat{X}_{\underline{s}}^{i,N}\right|^{2\ell+2}\right] + \frac{6}{\varepsilon_1}L^2\left[2^{2\ell-1}\left|\widehat{X}_s^{i,N} - \widehat{X}_{\underline{s}}^{i,N}\right|^{2\ell}\mathcal{W}_2^2\left(\mu_s^{\widehat{\boldsymbol{X}}^N}, \mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N}\right)\right. \\
&\quad \left. + (2^{2\ell-1}+1)\left|\widehat{X}_{\underline{s}}^{i,N}\right|^{2\ell}\left|\widehat{X}_s^{i,N} - \widehat{X}_{\underline{s}}^{i,N}\right|^2\right] + \frac{6}{\varepsilon_1}L^2(2^{2\ell-1}+1)\left|\widehat{X}_{\underline{s}}^{i,N}\right|^{2\ell}\mathcal{W}_2^2\left(\mu_s^{\widehat{\boldsymbol{X}}^N}, \mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N}\right).
\end{aligned} \tag{3.66}$$

Second, by using Cauchy's inequality, we have that for any $\varepsilon_2 > 0$,

$$\begin{aligned}
&\left|\sigma(X_s^{i,N}, \mu_s^{\boldsymbol{X}^N}) - \sigma_\Delta\left(\widehat{X}_{\underline{s}}^{i,N}, \mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N}\right)\right|^2 \\
&\leq (1+\varepsilon_2)\left|\sigma(X_s^{i,N}, \mu_s^{\boldsymbol{X}^N}) - \sigma\left(\widehat{X}_s^{i,N}, \mu_s^{\widehat{\boldsymbol{X}}^N}\right)\right|^2 \\
&\quad + 2\left(1 + \frac{1}{\varepsilon_2}\right)\left[\left|\sigma\left(\widehat{X}_s^{i,N}, \mu_s^{\widehat{\boldsymbol{X}}^N}\right) - \sigma\left(\widehat{X}_{\underline{s}}^{i,N}, \mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N}\right)\right|^2\right. \\
&\quad \left. + \left|\sigma\left(\widehat{X}_{\underline{s}}^{i,N}, \mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N}\right) - \sigma_\Delta\left(\widehat{X}_{\underline{s}}^{i,N}, \mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N}\right)\right|^2\right].
\end{aligned} \tag{3.67}$$

It follows from Remark 3.2.1 that

$$\begin{aligned}
&\left|\sigma\left(\widehat{X}_s^{i,N}, \mu_s^{\widehat{\boldsymbol{X}}^N}\right) - \sigma\left(\widehat{X}_{\underline{s}}^{i,N}, \mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N}\right)\right|^2 \\
&\leq \frac{L\widetilde{L}}{1+\varepsilon}\left(1 + \left|\widehat{X}_s^{i,N}\right|^\ell + \left|\widehat{X}_{\underline{s}}^{i,N}\right|^\ell\right)\left(\left|\widehat{X}_s^{i,N} - \widehat{X}_{\underline{s}}^{i,N}\right|^2 + \mathcal{W}_2^2\left(\mu_s^{\widehat{\boldsymbol{X}}^N}, \mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N}\right)\right)
\end{aligned}$$

$$
\begin{aligned}
&\le \frac{L\widetilde{L}}{1+\varepsilon}\left[1+2^{\ell-1}\Big(\left|\widehat{X}_s^{i,N}-\widehat{X}_{\underline{s}}^{i,N}\right|^{\ell}+\left|\widehat{X}_{\underline{s}}^{i,N}\right|^{\ell}\Big)+\left|\widehat{X}_{\underline{s}}^{i,N}\right|^{\ell}\right]\\
&\quad\times\left[\left|\widehat{X}_s^{i,N}-\widehat{X}_{\underline{s}}^{i,N}\right|^{2}+\mathcal{W}_2^2\left(\mu_s^{\widehat{\boldsymbol{X}}^N},\mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N}\right)\right]\\
&=\frac{L\widetilde{L}}{1+\varepsilon}\Bigg(\left|\widehat{X}_s^{i,N}-\widehat{X}_{\underline{s}}^{i,N}\right|^{2}+\mathcal{W}_2^2\left(\mu_s^{\widehat{\boldsymbol{X}}^N},\mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N}\right)+2^{\ell-1}\left|\widehat{X}_s^{i,N}-\widehat{X}_{\underline{s}}^{i,N}\right|^{\ell+2}\\
&\quad+2^{\ell-1}\left|\widehat{X}_s^{i,N}-\widehat{X}_{\underline{s}}^{i,N}\right|^{\ell}\mathcal{W}_2^2\left(\mu_s^{\widehat{\boldsymbol{X}}^N},\mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N}\right)+(2^{\ell-1}+1)\left|\widehat{X}_{\underline{s}}^{i,N}\right|^{\ell}\left|\widehat{X}_s^{i,N}-\widehat{X}_{\underline{s}}^{i,N}\right|^{2}\\
&\quad+(2^{\ell-1}+1)\big|\widehat{X}_{\underline{s}}^{i,N}\big|^{\ell}\mathcal{W}_2^2\left(\mu_s^{\widehat{\boldsymbol{X}}^N},\mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N}\right)\Bigg).
\end{aligned}
\tag{3.68}
$$

Then, it follows from **T6** and Remark 3.2.1(iii) that

$$
\begin{aligned}
&\left|\sigma\left(\widehat{X}_{\underline{s}}^{i,N},\mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N}\right)-\sigma_\Delta\left(\widehat{X}_{\underline{s}}^{i,N},\mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N}\right)\right|^2\le L_4^2\Delta\left|\sigma\left(\widehat{X}_{\underline{s}}^{i,N},\mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N}\right)\right|^4\left(1+\left|\widehat{X}_{\underline{s}}^{i,N}\right|\right)^2\\
&\le 4L_4^2\Delta\left[16\Big(\frac{L\widetilde{L}}{1+\varepsilon}\Big)^2\Big(1+|\widehat{X}_{\underline{s}}^{i,N}|^{2\ell}\Big)\Big(|\widehat{X}_{\underline{s}}^{i,N}|^4+\mathcal{W}_2^4\big(\mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N},\delta_0\big)\Big)+4\left|\sigma(0,\delta_0)\right|^4\right]\\
&\quad\times\Big(1+|\widehat{X}_{\underline{s}}^{i,N}|^2\Big).
\end{aligned}
\tag{3.69}
$$

Third, using Cauchy's inequality, we obtain that for any $\varepsilon_3>0$,

$$
\begin{aligned}
&\left|c(X_s^{i,N},\mu_s^{\boldsymbol{X}^N})-c_\Delta\left(\widehat{X}_{\underline{s}}^{i,N},\mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N}\right)\right|^2\\
&\le(1+\varepsilon_3)\left|c(X_s^{i,N},\mu_s^{\boldsymbol{X}^N})-c\left(\widehat{X}_s^{i,N},\mu_s^{\widehat{\boldsymbol{X}}^N}\right)\right|^2+2\left(1+\frac{1}{\varepsilon_3}\right)\\
&\quad\times\left[\left|c\left(\widehat{X}_s^{i,N},\mu_s^{\widehat{\boldsymbol{X}}^N}\right)-c\left(\widehat{X}_{\underline{s}}^{i,N},\mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N}\right)\right|^2+\left|c\left(\widehat{X}_{\underline{s}}^{i,N},\mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N}\right)-c_\Delta\left(\widehat{X}_{\underline{s}}^{i,N},\mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N}\right)\right|^2\right].
\end{aligned}
\tag{3.70}
$$

Thanks to Remark 3.2.1(ii), we have

$$
\begin{aligned}
&\left|c\left(\widehat{X}_s^{i,N},\mu_s^{\widehat{\boldsymbol{X}}^N}\right)-c\left(\widehat{X}_{\underline{s}}^{i,N},\mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N}\right)\right|^2\int_{\mathbb{R}_0^d}|z|^2\nu(dz)\\
&\le\frac{L\widetilde{L}}{1+\varepsilon}\left(1+\left|\widehat{X}_s^{i,N}\right|^{\ell}+\left|\widehat{X}_{\underline{s}}^{i,N}\right|^{\ell}\right)\left(\left|\widehat{X}_s^{i,N}-\widehat{X}_{\underline{s}}^{i,N}\right|^2+\mathcal{W}_2^2\left(\mu_s^{\widehat{\boldsymbol{X}}^N},\mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N}\right)\right)\\
&\le \frac{L\widetilde{L}}{1+\varepsilon}\left[1+2^{\ell-1}\Big(\left|\widehat{X}_s^{i,N}-\widehat{X}_{\underline{s}}^{i,N}\right|^{\ell}+\left|\widehat{X}_{\underline{s}}^{i,N}\right|^{\ell}\Big)+\left|\widehat{X}_{\underline{s}}^{i,N}\right|^{\ell}\right]\\
&\quad\times\left[\left|\widehat{X}_s^{i,N}-\widehat{X}_{\underline{s}}^{i,N}\right|^{2}+\mathcal{W}_2^2\left(\mu_s^{\widehat{\boldsymbol{X}}^N},\mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N}\right)\right]\\
&=\frac{L\widetilde{L}}{1+\varepsilon}\Bigg[\left|\widehat{X}_s^{i,N}-\widehat{X}_{\underline{s}}^{i,N}\right|^{2}+\mathcal{W}_2^2\left(\mu_s^{\widehat{\boldsymbol{X}}^N},\mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N}\right)+2^{\ell-1}\left|\widehat{X}_s^{i,N}-\widehat{X}_{\underline{s}}^{i,N}\right|^{\ell+2}
\end{aligned}
$$

$$+ 2^{\ell-1}\left|\widehat{X}_s^{i,N}-\widehat{X}_{\underline{s}}^{i,N}\right|^{\ell}\mathcal{W}_2^2\left(\mu_s^{\widehat{\boldsymbol{X}}^N},\mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N}\right)+(2^{\ell-1}+1)\left|\widehat{X}_{\underline{s}}^{i,N}\right|^{\ell}\left|\widehat{X}_s^{i,N}-\widehat{X}_{\underline{s}}^{i,N}\right|^2$$
$$+ (2^{\ell-1}+1)\big|\widehat{X}_{\underline{s}}^{i,N}\big|^{\ell}\mathcal{W}_2^2\left(\mu_s^{\widehat{\boldsymbol{X}}^N},\mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N}\right)\Bigg]. \tag{3.71}$$

Thanks to Condition **T6** and Remark 3.2.1(iii) and (i), we have

$$\left|c\left(\widehat{X}_{\underline{s}}^{i,N},\mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N}\right)-c_\Delta\left(\widehat{X}_{\underline{s}}^{i,N},\mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N}\right)\right|^2$$
$$\leq L_4^2\Delta\left|c\left(\widehat{X}_{\underline{s}}^{i,N},\mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N}\right)\right|^4\left(1+\left|\widehat{X}_{\underline{s}}^{i,N}\right|+\left|b\left(\widehat{X}_{\underline{s}}^{i,N},\mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N}\right)\right|\right)^2$$
$$\leq \frac{6L_4^2\Delta}{\left(\int_{\mathbb{R}_0^d}|z|^2\nu(dz)\right)^2}\left[16\Big(\frac{L\widetilde{L}}{1+\varepsilon}\Big)^2\Big(1+|\widehat{X}_{\underline{s}}^{i,N}|^{2\ell}\Big)\Big(|\widehat{X}_{\underline{s}}^{i,N}|^4+\mathcal{W}_2^4\big(\mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N},\delta_0\big)\Big)\right. \tag{3.72}$$
$$\left.+4\,|c(0,\delta_0)|^4\left(\int_{\mathbb{R}_0^d}|z|^2\nu(dz)\right)^2\right]\left[1+|\widehat{X}_{\underline{s}}^{i,N}|^2+2\Bigg(|b(0,\delta_0)|^2+4L^2\big(1+|\widehat{X}_{\underline{s}}^{i,N}|^{2\ell}\big)\right.$$
$$\left.\times\Big(|\widehat{X}_{\underline{s}}^{i,N}|^2+\mathcal{W}_2^2\big(\mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N},\delta_0\big)\Big)\Bigg)\right]. \tag{3.73}$$

Therefore, inserting all estimations (3.65) – (3.73) into (3.64), and choosing $\varepsilon_2=\varepsilon_3=\varepsilon$, we obtain that for any $\lambda\in\mathbb{R}$ and $\varepsilon_1>0$,

$$e^{-\lambda t}|X_t^{i,N}-\widehat{X}_t^{i,N}|^2$$
$$\leq\int_0^t e^{-\lambda s}\Bigg\{-\lambda|X_s^{i,N}-\widehat{X}_s^{i,N}|^2+\varepsilon_1\left|X_s^{i,N}-\widehat{X}_s^{i,N}\right|^2$$
$$+2\left\langle X_s^{i,N}-\widehat{X}_s^{i,N},b(X_s^{i,N},\mu_s^{\boldsymbol{X}^N})-b\left(\widehat{X}_s^{i,N},\mu_s^{\widehat{\boldsymbol{X}}^N}\right)\right\rangle$$
$$+(1+\varepsilon)\left|\sigma(X_s^{i,N},\mu_s^{\boldsymbol{X}^N})-\sigma\left(\widehat{X}_s^{i,N},\mu_s^{\widehat{\boldsymbol{X}}^N}\right)\right|^2$$
$$+(1+\varepsilon)\left|c(X_s^{i,N},\mu_s^{\boldsymbol{X}^N})-c\left(\widehat{X}_s^{i,N},\mu_s^{\widehat{\boldsymbol{X}}^N}\right)\right|^2\int_{\mathbb{R}_0^d}|z|^2\nu(dz)$$
$$+\frac{6}{\varepsilon_1}L^2\Bigg[\left|\widehat{X}_s^{i,N}-\widehat{X}_{\underline{s}}^{i,N}\right|^2+\mathcal{W}_2^2\left(\mu_s^{\widehat{\boldsymbol{X}}^N},\mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N}\right)+2^{2\ell-1}\left|\widehat{X}_s^{i,N}-\widehat{X}_{\underline{s}}^{i,N}\right|^{2\ell+2}$$
$$+2^{2\ell-1}\left|\widehat{X}_s^{i,N}-\widehat{X}_{\underline{s}}^{i,N}\right|^{2\ell}\mathcal{W}_2^2\left(\mu_s^{\widehat{\boldsymbol{X}}^N},\mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N}\right)+(2^{2\ell-1}+1)\left|\widehat{X}_{\underline{s}}^{i,N}\right|^{2\ell}\left|\widehat{X}_s^{i,N}-\widehat{X}_{\underline{s}}^{i,N}\right|^2$$
$$+(2^{2\ell-1}+1)\left|\widehat{X}_{\underline{s}}^{i,N}\right|^{2\ell}\mathcal{W}_2^2\left(\mu_s^{\widehat{\boldsymbol{X}}^N},\mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N}\right)\Bigg]+2\big(1+\frac{1}{\varepsilon}\big)\Bigg[\frac{L\widetilde{L}}{1+\varepsilon}\Bigg(\left|\widehat{X}_s^{i,N}-\widehat{X}_{\underline{s}}^{i,N}\right|^2$$
$$+\mathcal{W}_2^2\left(\mu_s^{\widehat{\boldsymbol{X}}^N},\mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N}\right)+2^{\ell-1}\left|\widehat{X}_s^{i,N}-\widehat{X}_{\underline{s}}^{i,N}\right|^{\ell+2}+2^{\ell-1}\left|\widehat{X}_s^{i,N}-\widehat{X}_{\underline{s}}^{i,N}\right|^{\ell}\mathcal{W}_2^2\left(\mu_s^{\widehat{\boldsymbol{X}}^N},\mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N}\right)$$

$$
\begin{aligned}
&+(2^{\ell-1}+1)\left|\widehat{X}_{\underline{s}}^{i,N}\right|^{\ell}\left|\widehat{X}_{s}^{i,N}-\widehat{X}_{\underline{s}}^{i,N}\right|^{2}+(2^{\ell-1}+1)\big|\widehat{X}_{\underline{s}}^{i,N}\big|^{\ell}\mathcal{W}_2^2\left(\mu_s^{\widehat{\boldsymbol{X}}^N},\mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N}\right)\Bigg)\\
&+4L_4^2\Delta\Bigg(16\Big(\frac{L\widetilde{L}}{1+\varepsilon}\Big)^2\Big(1+|\widehat{X}_{\underline{s}}^{i,N}|^{2\ell}\Big)\Big(|\widehat{X}_{\underline{s}}^{i,N}|^4+\mathcal{W}_2^4\big(\mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N},\delta_0\big)\Big)+4\,|\sigma(0,\delta_0)|^4\Bigg)\\
&\times\Big(1+|\widehat{X}_{\underline{s}}^{i,N}|^2\Big)\Bigg]+2\left(1+\frac{1}{\varepsilon}\right)\Bigg[\frac{L\widetilde{L}}{1+\varepsilon}\Bigg(\left|\widehat{X}_{s}^{i,N}-\widehat{X}_{\underline{s}}^{i,N}\right|^{2}+\mathcal{W}_2^2\left(\mu_s^{\widehat{\boldsymbol{X}}^N},\mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N}\right)\\
&+2^{\ell-1}\left|\widehat{X}_{s}^{i,N}-\widehat{X}_{\underline{s}}^{i,N}\right|^{\ell+2}+2^{\ell-1}\left|\widehat{X}_{s}^{i,N}-\widehat{X}_{\underline{s}}^{i,N}\right|^{\ell}\ \mathcal{W}_2^2\left(\mu_s^{\widehat{\boldsymbol{X}}^N},\mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N}\right)\\
&+(2^{\ell-1}+1)\left|\widehat{X}_{\underline{s}}^{i,N}\right|^{\ell}\left|\widehat{X}_{s}^{i,N}-\widehat{X}_{\underline{s}}^{i,N}\right|^{2}+(2^{\ell-1}+1)\big|\widehat{X}_{\underline{s}}^{i,N}\big|^{\ell}\mathcal{W}_2^2\left(\mu_s^{\widehat{\boldsymbol{X}}^N},\mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N}\right)\Bigg)\\
&+\frac{6L_4^2\Delta}{\int_{\mathbb{R}_0^d}|z|^2\nu(dz)}\Bigg(16\Big(\frac{L\widetilde{L}}{1+\varepsilon}\Big)^2\Big(1+|\widehat{X}_{\underline{s}}^{i,N}|^{2\ell}\Big)\Big(|\widehat{X}_{\underline{s}}^{i,N}|^4+\mathcal{W}_2^4\big(\mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N},\delta_0\big)\Big)\\
&+4\,|c(0,\delta_0)|^4\left(\int_{\mathbb{R}_0^d}|z|^2\nu(dz)\right)^2\Bigg)\Bigg(1+|\widehat{X}_{\underline{s}}^{i,N}|^2+2\Bigg(|b(0,\delta_0)|^2+4L^2\left(1+|\widehat{X}_{\underline{s}}^{i,N}|^{2\ell}\right)\\
&\times\left(|\widehat{X}_{\underline{s}}^{i,N}|^2+\mathcal{W}_2^2\big(\mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N},\delta_0\big)\right)\Bigg)\Bigg)\Bigg]\Bigg\}ds\\
&+2\int_0^t e^{-\lambda s}\left\langle X_s^{i,N}-\widehat{X}_s^{i,N},\big(\sigma(X_s^{i,N},\mu_s^{\boldsymbol{X}^N})-\sigma_\Delta(\widehat{X}_{\underline{s}}^{i,N},\mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N})\big)dW_s^i\right\rangle\\
&+\int_0^t\int_{\mathbb{R}_0^d}e^{-\lambda s}\Bigg(|X_{s-}^{i,N}-\widehat{X}_{s-}^{i,N}+\big(c(X_{s-}^{i,N},\mu_{s-}^{\boldsymbol{X}^N})-c_\Delta(\widehat{X}_{\underline{s}-}^{i,N},\mu_{\underline{s}-}^{\widehat{\boldsymbol{X}}^N})\big)z|^2\\
&-|X_{s-}^{i,N}-\widehat{X}_{s-}^{i,N}|^2\Bigg)\widetilde{N}^i(ds,dz).
\end{aligned}
$$

Using Condition **C2** for $\kappa_1=\kappa_2=1+\varepsilon$, $L_1\in\mathbb{R}$, $L_2\geq 0$, we obtain that for any $\lambda\in\mathbb{R}$ and $\varepsilon_1>0$,

$$
\begin{aligned}
&e^{-\lambda t}|X_t^{i,N}-\widehat{X}_t^{i,N}|^2\\
\leq&\int_0^t e^{-\lambda s}\Bigg\{-\lambda|X_s^{i,N}-\widehat{X}_s^{i,N}|^2+\varepsilon_1\left|X_s^{i,N}-\widehat{X}_s^{i,N}\right|^2+L_1|X_s^{i,N}-\widehat{X}_s^{i,N}|^2\\
&+L_2\mathcal{W}_2^2\left(\mu_s^{\boldsymbol{X}^N},\mu_s^{\widehat{\boldsymbol{X}}^N}\right)+\frac{6}{\varepsilon_1}L^2\Bigg[\left|\widehat{X}_{s}^{i,N}-\widehat{X}_{\underline{s}}^{i,N}\right|^{2}+\mathcal{W}_2^2\left(\mu_s^{\widehat{\boldsymbol{X}}^N},\mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N}\right)\\
&+2^{2\ell-1}\left|\widehat{X}_{s}^{i,N}-\widehat{X}_{\underline{s}}^{i,N}\right|^{2\ell+2}+2^{2\ell-1}\left|\widehat{X}_{s}^{i,N}-\widehat{X}_{\underline{s}}^{i,N}\right|^{2\ell}\mathcal{W}_2^2\left(\mu_s^{\widehat{\boldsymbol{X}}^N},\mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N}\right)\\
&+(2^{2\ell-1}+1)\left|\widehat{X}_{\underline{s}}^{i,N}\right|^{2\ell}\left|\widehat{X}_{s}^{i,N}-\widehat{X}_{\underline{s}}^{i,N}\right|^{2}+(2^{2\ell-1}+1)\left|\widehat{X}_{\underline{s}}^{i,N}\right|^{2\ell}\mathcal{W}_2^2\left(\mu_s^{\widehat{\boldsymbol{X}}^N},\mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N}\right)\Bigg]
\end{aligned}
$$

$$
\begin{aligned}
&+2\left(1+\frac{1}{\varepsilon}\right)\Bigg[\frac{L\widetilde{L}}{1+\varepsilon}\Bigg(\left|\widehat{X}^{i,N}_s-\widehat{X}^{i,N}_{\underline{s}}\right|^2+\mathcal{W}_2^2\left(\mu_s^{\widehat{\boldsymbol{X}}^N},\mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N}\right)+2^{\ell-1}\left|\widehat{X}^{i,N}_s-\widehat{X}^{i,N}_{\underline{s}}\right|^{\ell+2}\\
&+2^{\ell-1}\left|\widehat{X}^{i,N}_s-\widehat{X}^{i,N}_{\underline{s}}\right|^{\ell}\mathcal{W}_2^2\left(\mu_s^{\widehat{\boldsymbol{X}}^N},\mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N}\right)+(2^{\ell-1}+1)\left|\widehat{X}^{i,N}_{\underline{s}}\right|^{\ell}\left|\widehat{X}^{i,N}_s-\widehat{X}^{i,N}_{\underline{s}}\right|^2\\
&+(2^{\ell-1}+1)\big|\widehat{X}^{i,N}_{\underline{s}}\big|^{\ell}\mathcal{W}_2^2\left(\mu_s^{\widehat{\boldsymbol{X}}^N},\mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N}\right)\Bigg)+4L_4^2\Delta\Bigg(16\Big(\frac{L\widetilde{L}}{1+\varepsilon}\Big)^2\Big(1+|\widehat{X}^{i,N}_{\underline{s}}|^{2\ell}\Big)\\
&\times\Big(|\widehat{X}^{i,N}_{\underline{s}}|^4+\mathcal{W}_2^4\big(\mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N},\delta_0\big)\Big)+4\,|\sigma(0,\delta_0)|^4\Bigg)\Big(1+|\widehat{X}^{i,N}_{\underline{s}}|^2\Big)\Bigg]\\
&+2\left(1+\frac{1}{\varepsilon}\right)\Bigg[\frac{L\widetilde{L}}{1+\varepsilon}\Bigg(\left|\widehat{X}^{i,N}_s-\widehat{X}^{i,N}_{\underline{s}}\right|^2+\mathcal{W}_2^2\left(\mu_s^{\widehat{\boldsymbol{X}}^N},\mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N}\right)+2^{\ell-1}\left|\widehat{X}^{i,N}_s-\widehat{X}^{i,N}_{\underline{s}}\right|^{\ell+2}\\
&+2^{\ell-1}\left|\widehat{X}^{i,N}_s-\widehat{X}^{i,N}_{\underline{s}}\right|^{\ell}\mathcal{W}_2^2\left(\mu_s^{\widehat{\boldsymbol{X}}^N},\mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N}\right)+(2^{\ell-1}+1)\left|\widehat{X}^{i,N}_{\underline{s}}\right|^{\ell}\left|\widehat{X}^{i,N}_s-\widehat{X}^{i,N}_{\underline{s}}\right|^2\\
&+(2^{\ell-1}+1)\big|\widehat{X}^{i,N}_{\underline{s}}\big|^{\ell}\mathcal{W}_2^2\left(\mu_s^{\widehat{\boldsymbol{X}}^N},\mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N}\right)\Bigg)+\frac{6L_4^2\Delta}{\int_{\mathbb{R}^d_0}|z|^2\nu(dz)}\Bigg(16\Big(\frac{L\widetilde{L}}{1+\varepsilon}\Big)^2\Big(1+|\widehat{X}^{i,N}_{\underline{s}}|^{2\ell}\Big)\\
&\times\Big(|\widehat{X}^{i,N}_{\underline{s}}|^4+\mathcal{W}_2^4\big(\mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N},\delta_0\big)\Big)+4\,|c(0,\delta_0)|^4\Big(\int_{\mathbb{R}^d_0}|z|^2\nu(dz)\Big)^2\Bigg)\Bigg(1+|\widehat{X}^{i,N}_{\underline{s}}|^2\\
&+2\Bigg(|b(0,\delta_0)|^2+4L^2\Big(1+|\widehat{X}^{i,N}_{\underline{s}}|^{2\ell}\Big)\Big(|\widehat{X}^{i,N}_{\underline{s}}|^2+\mathcal{W}_2^2\big(\mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N},\delta_0\big)\Big)\Bigg)\Bigg)\Bigg]\Bigg\}ds\\
&+2\int_0^t e^{-\lambda s}\Big\langle X^{i,N}_s-\widehat{X}^{i,N}_s,\big(\sigma(X^{i,N}_s,\mu_s^{\boldsymbol{X}^N})-\sigma_\Delta(\widehat{X}^{i,N}_{\underline{s}},\mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N})\big)dW^i_s\Big\rangle\\
&+\int_0^t\int_{\mathbb{R}^d_0}e^{-\lambda s}\Bigg(|X^{i,N}_{s-}-\widehat{X}^{i,N}_{s-}+\Big(c(X^{i,N}_{s-},\mu_{s-}^{\boldsymbol{X}^N})-c_\Delta(\widehat{X}^{i,N}_{\underline{s}-},\mu_{\underline{s}-}^{\widehat{\boldsymbol{X}}^N})\Big)z|^2\\
&-|X^{i,N}_{s-}-\widehat{X}^{i,N}_{s-}|^2\Bigg)\widetilde{N}^i(ds,dz). \qquad (3.74)
\end{aligned}
$$

Now, using Lemma 3.7.1 and Proposition 3.6.9, we have the following estimates

$$
\begin{aligned}
&\mathbb{E}\left[\mathcal{W}_2^2\big(\mu_s^{\widehat{\boldsymbol{X}}^N},\mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N}\big)\right]\le\frac{1}{N}\sum_{j=1}^N\mathbb{E}\left[\left|\widehat{X}^{j,N}_s-\widehat{X}^{j,N}_{\underline{s}}\right|^2\right]=\mathbb{E}\left[\left|\widehat{X}^{i,N}_s-\widehat{X}^{i,N}_{\underline{s}}\right|^2\right]\le C\Delta,\\
&\mathbb{E}\left[\mathcal{W}_2^2\big(\mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N},\delta_0\big)\right]=\mathbb{E}\left[\frac{1}{N}\sum_{j=1}^N\left|\widehat{X}^{j,N}_{\underline{s}}\right|^2\right]=\frac{1}{N}\sum_{j=1}^N\mathbb{E}\left[\left|\widehat{X}^{j,N}_{\underline{s}}\right|^2\right]=\mathbb{E}\left[\left|\widehat{X}^{i,N}_{\underline{s}}\right|^2\right]\le C,\\
&\mathbb{E}\left[\mathcal{W}_2^4\big(\mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N},\delta_0\big)\right]=\mathbb{E}\left[\left(\frac{1}{N}\sum_{j=1}^N\left|\widehat{X}^{j,N}_{\underline{s}}\right|^2\right)^2\right]\le\frac{1}{N}\sum_{j=1}^N\mathbb{E}\left[\left|\widehat{X}^{j,N}_{\underline{s}}\right|^4\right]=\mathbb{E}\left[\left|\widehat{X}^{i,N}_{\underline{s}}\right|^4\right]\le C,
\end{aligned}
\qquad (3.75)
$$

for any $i \in \{1, \dots, N\}$ and some constant $C > 0$.

Using the estimate

$$\mathcal{W}_2^p\left(\mu_s^{\widehat{\boldsymbol{X}}^N}, \mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N}\right) \le \frac{1}{N}\sum_{j=1}^{N}\left|\widehat{X}_s^{j,N} - \widehat{X}_{\underline{s}}^{j,N}\right|^p,$$

valid for any $p \ge 2$ and Lemma 3.7.1, we have that for $\rho \in \{\ell, 2\ell\}$,

$$\begin{aligned}
&\mathbb{E}\left[\left|\widehat{X}_s^{i,N} - \widehat{X}_{\underline{s}}^{i,N}\right|^\rho \mathcal{W}_2^2\left(\mu_s^{\widehat{\boldsymbol{X}}^N}, \mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N}\right)\right] \le \frac{1}{N}\sum_{j=1}^{N}\mathbb{E}\left[\left|\widehat{X}_s^{i,N} - \widehat{X}_{\underline{s}}^{i,N}\right|^\rho \left|\widehat{X}_s^{j,N} - \widehat{X}_{\underline{s}}^{j,N}\right|^2\right] \\
&= \frac{1}{N}\mathbb{E}\left[\left|\widehat{X}_s^{i,N} - \widehat{X}_{\underline{s}}^{i,N}\right|^{\rho+2}\right] + \frac{1}{N}\sum_{j\neq i}\mathbb{E}\left[\left|\widehat{X}_s^{i,N} - \widehat{X}_{\underline{s}}^{i,N}\right|^\rho\right]\mathbb{E}\left[\left|\widehat{X}_s^{j,N} - \widehat{X}_{\underline{s}}^{j,N}\right|^2\right] \\
&\le C\Delta.
\end{aligned} \tag{3.76}$$

Next, using Lemma 3.7.1, Proposition 3.6.9 and the fact that $p_0 \ge 4\ell + 6$, we get that

$$\begin{aligned}
&\mathbb{E}\left[\left|\widehat{X}_s^{i,N} - \widehat{X}_{\underline{s}}^{i,N}\right|^q\right] \le C\Delta, \\
&\mathbb{E}\left[\left|\widehat{X}_s^{i,N} - \widehat{X}_{\underline{s}}^{i,N}\right|^\ell\right] \le \left(\mathbb{E}\left[\left|\widehat{X}_s^{i,N} - \widehat{X}_{\underline{s}}^{i,N}\right|^{2\ell}\right]\right)^{1/2} \le C\sqrt{\Delta} \le C, \\
&\mathbb{E}\left[\left|\widehat{X}_{\underline{s}}^{i,N}\right|^\upsilon\right] \le C,
\end{aligned} \tag{3.77}$$

for $q \in \{2; \ell + 2; 2\ell; 2\ell + 2\}$, $\upsilon \in \{\ell; 2\ell; 2\ell + 4; 2\ell + 6; 4\ell + 6\}$, and

$$\begin{aligned}
\mathbb{E}\left[\left|\widehat{X}_{\underline{s}}^{i,N}\right|^\kappa \left|\widehat{X}_s^{i,N} - \widehat{X}_{\underline{s}}^{i,N}\right|^2\right] &= \mathbb{E}\left[\mathbb{E}\left[\left|\widehat{X}_{\underline{s}}^{i,N}\right|^\kappa \left|\widehat{X}_s^{i,N} - \widehat{X}_{\underline{s}}^{i,N}\right|^2 \big| \mathcal{F}_{\underline{s}}\right]\right] \\
&= \mathbb{E}\left[\left|\widehat{X}_{\underline{s}}^{i,N}\right|^\kappa \mathbb{E}\left[\left|\widehat{X}_s^{i,N} - \widehat{X}_{\underline{s}}^{i,N}\right|^2 \big| \mathcal{F}_{\underline{s}}\right]\right] \\
&\le C\Delta\mathbb{E}\left[\left|\widehat{X}_{\underline{s}}^{i,N}\right|^\kappa\right] \\
&\le C\Delta,
\end{aligned} \tag{3.78}$$

for $\kappa \in \{\ell; 2\ell\}$ and some constant $C > 0$. Furthermore, using Lemma 3.7.1 and Proposition 3.6.9, we obtain that for $\varrho \in \{\ell, 2\ell\}$,

$$\begin{aligned}
&\mathbb{E}\left[\left|\widehat{X}_{\underline{s}}^{i,N}\right|^\varrho \mathcal{W}_2^2\left(\mu_s^{\widehat{\boldsymbol{X}}^N}, \mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N}\right)\right] \le \frac{1}{N}\sum_{j=1}^{N}\mathbb{E}\left[\left|\widehat{X}_{\underline{s}}^{i,N}\right|^\varrho \left|\widehat{X}_s^{j,N} - \widehat{X}_{\underline{s}}^{j,N}\right|^2\right] \\
&= \frac{1}{N}\mathbb{E}\left[\left|\widehat{X}_{\underline{s}}^{i,N}\right|^\varrho \left|\widehat{X}_s^{i,N} - \widehat{X}_{\underline{s}}^{i,N}\right|^2\right] + \frac{1}{N}\sum_{j\neq i}\mathbb{E}\left[\left|\widehat{X}_{\underline{s}}^{i,N}\right|^\varrho\right]\mathbb{E}\left[\left|\widehat{X}_s^{j,N} - \widehat{X}_{\underline{s}}^{j,N}\right|^2\right] \\
&\le C\Delta.
\end{aligned} \tag{3.79}$$

Using Proposition 3.6.9, we obtain that for $\vartheta \in \{2\ell+2; 4\ell+2\}$,

$$\mathbb{E}\left[\left|\widehat{X}^{i,N}_{\underline{s}}\right|^{\vartheta} \mathcal{W}_2^4\big(\mu^{\widehat{\boldsymbol{X}}^N}_{\underline{s}}, \delta_0\big)\right] \leq \frac{1}{N}\sum_{j=1}^{N} \mathbb{E}\left[\left|\widehat{X}^{i,N}_{\underline{s}}\right|^{\vartheta}\left|\widehat{X}^{j,N}_{\underline{s}}\right|^{4}\right]$$

$$= \frac{1}{N}\mathbb{E}\left[\left|\widehat{X}^{i,N}_{\underline{s}}\right|^{\vartheta+4}\right] + \frac{1}{N}\sum_{j\neq i}\mathbb{E}\left[\left|\widehat{X}^{i,N}_{\underline{s}}\right|^{\vartheta}\right]\mathbb{E}\left[\left|\widehat{X}^{j,N}_{\underline{s}}\right|^{4}\right]$$

$$\leq C. \tag{3.80}$$

Proceeding similarly to the above, we get that

$$\mathbb{E}\left[\left|\widehat{X}^{i,N}_{\underline{s}}\right|^{4\ell} \mathcal{W}_2^2\big(\mu^{\widehat{\boldsymbol{X}}^N}_{\underline{s}}, \delta_0\big)\mathcal{W}_2^4\big(\mu^{\widehat{\boldsymbol{X}}^N}_{\underline{s}}, \delta_0\big)\right] = \mathbb{E}\left[\left|\widehat{X}^{i,N}_{\underline{s}}\right|^{4\ell} \mathcal{W}_2^6\big(\mu^{\widehat{\boldsymbol{X}}^N}_{\underline{s}}, \delta_0\big)\right] \leq C. \tag{3.81}$$

Consequently, plugging (3.75)-(3.81) into (3.74), taking the expectation on both sides and choosing $\lambda = \varepsilon_1 + L_1 + L_2$, we get that for any $t \in [0, T]$,

$$\mathbb{E}\left[e^{-(\varepsilon_1+L_1+L_2)t}\left|X^{i,N}_t - \widehat{X}^{i,N}_t\right|^2\right] \leq C\Delta\int_0^t e^{-(\varepsilon_1+L_1+L_2)s}ds.$$

This implies that

$$\max_{i\in\{1,\ldots,N\}}\sup_{t\in[0,T]}\mathbb{E}\left[|X^{i,N}_t - \widehat{X}^{i,N}_t|^2\right] \leq C_T\Delta, \tag{3.82}$$

for some positive constant $C_T = C(x_0, L, L_1, L_2, \widetilde{\gamma}_1, \widetilde{\gamma}_2, \widetilde{\eta}, \widetilde{L_3}, T)$, which shows (3.61).

Next, for any stopping time $\tau \leq T$, using again (3.74) with $t = \tau$, $\lambda = \epsilon_1 + L_1$, taking the expectation of the above inequality on both sides and using again the estimates (3.75)–(3.81) and (3.82), we obtain that

$$\begin{aligned}
&\mathbb{E}\left[e^{-(\epsilon_1+L_1)\tau}|X^{i,N}_\tau - \widehat{X}^{i,N}_\tau|^2\right] \\
&\leq \int_0^T e^{-(\epsilon_1+L_1)s}\mathbb{E}\Bigg\{L_2\mathcal{W}_2^2\left(\mu^{\boldsymbol{X}^N}_s, \mu^{\widehat{\boldsymbol{X}}^N}_s\right) + \frac{6}{\varepsilon_1}L^2\Bigg[\left|\widehat{X}^{i,N}_s - \widehat{X}^{i,N}_{\underline{s}}\right|^2 + \mathcal{W}_2^2\left(\mu^{\widehat{\boldsymbol{X}}^N}_s, \mu^{\widehat{\boldsymbol{X}}^N}_{\underline{s}}\right) \\
&\quad + 2^{2\ell-1}\left|\widehat{X}^{i,N}_s - \widehat{X}^{i,N}_{\underline{s}}\right|^{2\ell+2} + 2^{2\ell-1}\left|\widehat{X}^{i,N}_s - \widehat{X}^{i,N}_{\underline{s}}\right|^{2\ell}\mathcal{W}_2^2\left(\mu^{\widehat{\boldsymbol{X}}^N}_s, \mu^{\widehat{\boldsymbol{X}}^N}_{\underline{s}}\right) \\
&\quad + (2^{2\ell-1}+1)\left|\widehat{X}^{i,N}_{\underline{s}}\right|^{2\ell}\left|\widehat{X}^{i,N}_s - \widehat{X}^{i,N}_{\underline{s}}\right|^2 + (2^{2\ell-1}+1)\left|\widehat{X}^{i,N}_{\underline{s}}\right|^{2\ell}\mathcal{W}_2^2\left(\mu^{\widehat{\boldsymbol{X}}^N}_s, \mu^{\widehat{\boldsymbol{X}}^N}_{\underline{s}}\right)\Bigg] \\
&\quad + 2\big(1+\frac{1}{\varepsilon}\big)\Bigg[\frac{L\widetilde{L}}{1+\varepsilon}\Bigg(\left|\widehat{X}^{i,N}_s - \widehat{X}^{i,N}_{\underline{s}}\right|^2 + \mathcal{W}_2^2\left(\mu^{\widehat{\boldsymbol{X}}^N}_s, \mu^{\widehat{\boldsymbol{X}}^N}_{\underline{s}}\right) + 2^{\ell-1}\left|\widehat{X}^{i,N}_s - \widehat{X}^{i,N}_{\underline{s}}\right|^{\ell+2} \\
&\quad + 2^{\ell-1}\left|\widehat{X}^{i,N}_s - \widehat{X}^{i,N}_{\underline{s}}\right|^{\ell}\mathcal{W}_2^2\left(\mu^{\widehat{\boldsymbol{X}}^N}_s, \mu^{\widehat{\boldsymbol{X}}^N}_{\underline{s}}\right) + (2^{\ell-1}+1)\left|\widehat{X}^{i,N}_{\underline{s}}\right|^{\ell}\left|\widehat{X}^{i,N}_s - \widehat{X}^{i,N}_{\underline{s}}\right|^2
\end{aligned}$$

$$
\begin{aligned}
&+ (2^{\ell-1}+1)\big|\widehat{X}^{i,N}_{\underline{s}}\big|^{\ell}\mathcal{W}_2^2\left(\mu_s^{\widehat{\boldsymbol{X}}^N},\mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N}\right)\Bigg) + 4L_4^2\Delta\Bigg(16\Big(\frac{L\widetilde{L}}{1+\varepsilon}\Big)^2\Big(1+|\widehat{X}^{i,N}_{\underline{s}}|^{2\ell}\Big)\\
&\times\Big(|\widehat{X}^{i,N}_{\underline{s}}|^4+\mathcal{W}_2^4\big(\mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N},\delta_0\big)\Big)+4\,|\sigma(0,\delta_0)|^4\Bigg)\Big(1+|\widehat{X}^{i,N}_{\underline{s}}|^2\Big)\Bigg]\\
&+2\left(1+\frac{1}{\varepsilon}\right)\Bigg[\frac{L\widetilde{L}}{1+\varepsilon}\Bigg(\Big|\widehat{X}^{i,N}_s-\widehat{X}^{i,N}_{\underline{s}}\Big|^2+\mathcal{W}_2^2\left(\mu_s^{\widehat{\boldsymbol{X}}^N},\mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N}\right)+2^{\ell-1}\Big|\widehat{X}^{i,N}_s-\widehat{X}^{i,N}_{\underline{s}}\Big|^{\ell+2}\\
&+2^{\ell-1}\Big|\widehat{X}^{i,N}_s-\widehat{X}^{i,N}_{\underline{s}}\Big|^{\ell}\mathcal{W}_2^2\left(\mu_s^{\widehat{\boldsymbol{X}}^N},\mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N}\right)+(2^{\ell-1}+1)\Big|\widehat{X}^{i,N}_{\underline{s}}\Big|^{\ell}\Big|\widehat{X}^{i,N}_s-\widehat{X}^{i,N}_{\underline{s}}\Big|^2\\
&+(2^{\ell-1}+1)\big|\widehat{X}^{i,N}_{\underline{s}}\big|^{\ell}\mathcal{W}_2^2\left(\mu_s^{\widehat{\boldsymbol{X}}^N},\mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N}\right)\Bigg)+\frac{6L_4^2\Delta}{\int_{\mathbb{R}_0^d}|z|^2\nu(dz)}\Bigg(16\Big(\frac{L\widetilde{L}}{1+\varepsilon}\Big)^2\Big(1+|\widehat{X}^{i,N}_{\underline{s}}|^{2\ell}\Big)\\
&\times\Big(|\widehat{X}^{i,N}_{\underline{s}}|^4+\mathcal{W}_2^4\big(\mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N},\delta_0\big)\Big)+4\,|c(0,\delta_0)|^4\left(\int_{\mathbb{R}_0^d}|z|^2\nu(dz)\right)^2\Bigg)\Bigg(1+|\widehat{X}^{i,N}_{\underline{s}}|^2\\
&+2\left(|b(0,\delta_0)|^2+4L^2\left(1+|\widehat{X}^{i,N}_{\underline{s}}|^{2\ell}\right)\left(|\widehat{X}^{i,N}_{\underline{s}}|^2+\mathcal{W}_2^2\big(\mu_{\underline{s}}^{\widehat{\boldsymbol{X}}^N},\delta_0\big)\right)\right)\Bigg)\Bigg]\Bigg\}ds\\
\le{}& \widetilde{C}_T\Delta,
\end{aligned}
$$

for some constant $\widetilde{C}_T>0$.

Therefore, due to Proposition IV.4.7 in [78], we get that for any $p\in(0,2)$,

$$
\max_{i\in\{1,\dots,N\}}\mathbb{E}\left[\sup_{t\in[0,T]}e^{-\frac{p(\epsilon_1+L_1)t}{2}}|X^{i,N}_t-\widehat{X}^{i,N}_t|^p\right]\le\left(\frac{2-p/2}{1-p/2}\right)(\widetilde{C}_T\Delta)^{p/2},
$$

which, combined with the fact that $e^{-\frac{p(\epsilon_1+L_1)t}{2}}\ge e^{-\frac{p|\epsilon_1+L_1|T}{2}}$, concludes (3.62).

Moreover, when $L_1+L_2<0$, we can always choose $\varepsilon_1>0$ such that $L_1+L_2+\varepsilon_1<0$. Consequently, when $L_1+L_2<0$ and $\widetilde{\gamma}<0$, the constant C_T in (3.82) now does not depend on T. Therefore, we have shown (3.63), which finishes the desired proof. □

We now state our main result on strong convergence in both finite and infinite time intervals of the tamed-adaptive Euler-Maruyama scheme for multidimensional McKean-Vlasov SDEs driven by Lévy processes.

Theorem 3.7.4. *Assume Conditions **C1**, **C3**–**C7**, **T2**–**T6** hold, and $p_0\ge 4\ell+6$. Assume further that there exists a constant $\varepsilon>0$ such that **C2** holds for $\kappa_1=\kappa_2=1+\varepsilon$, $L_1\in\mathbb{R}$, $L_2\ge 0$. Then for any $T>0$, there exists a positive constant $C_T=$*

$C(x_0, L, L_1, L_2, L_3, L_4, \widetilde{\gamma}_1, \widetilde{\gamma}_2, \widetilde{\eta}, \widetilde{L_3}, T)$ *such that*

$$\max_{i\in\{1,\dots,N\}} \sup_{t\in[0,T]} \mathbb{E}\left[\left|X_t^i - \widehat{X}_t^{i,N}\right|^2\right] \leq C_T\left(\Delta + \varphi(N)\right), \tag{3.83}$$

for any $N \in \mathbb{N}$, *where the constant* $C_T > 0$ *does not depend on* N.

Moreover, assume that $\gamma = \gamma_1 + \gamma_2 < 0$, $\widetilde{\gamma} = \widetilde{\gamma}_1 + \widetilde{\gamma}_2 < 0$ *and* $L_1 + L_2 < 0$. *Then, there exists a positive constant*
$C'' = C''(x_0, L, L_1, L_2, L_3, L_4, \widetilde{\gamma}_1, \widetilde{\gamma}_2, \widetilde{\eta}, \widetilde{L_3})$ *which does not depend on* T *such that*

$$\max_{i\in\{1,\dots,N\}} \sup_{t\geq 0} \mathbb{E}\left[\left|X_t^i - \widehat{X}_t^{i,N}\right|^2\right] \leq C''\left(\Delta + \varphi(N)\right). \tag{3.84}$$

Proof. As a consequence of Proposition 3.4.2 and Theorem 3.7.3, the proof is straightforward. Thus, we omit it. □

3.8 Numerical experiments

In this section, we consider the rate of convergence of the tamed-adaptive Euler-Maruyama scheme (3.21),(3.22), (3.59), (3.60) in Theorem 3.7.3 for fixed large values of N. We consider the following Lévy-driven McKean-Vlasov stochastic differential equation

$$\begin{aligned} dX_t &= \left(-1 - 3\left(X_t + \mathbb{E}\left[X_t\right]\right) - X_t|X_t|^{0.3}\right)dt + 0.2\left(1 + |X_t|^{1.1} + \mathbb{E}\left[X_t\right]\right)dW_t \\ &\quad + 0.2\left(X_{t-} + \mathbb{E}\left[X_{t-}\right]\right)dZ_t. \end{aligned} \tag{3.85}$$

That is,

$$b(x,\mu) = -1 - 3\left(x + \int_{\mathbb{R}} z\mu(dz)\right) - x|x|^{0.3},$$
$$\sigma(x,\mu) = 0.2\left(1 + |x|^{1.1} + \int_{\mathbb{R}} z\mu(dz)\right), \qquad c(x,\mu) = 0.2\left(x + \int_{\mathbb{R}} z\mu(dz)\right),$$

for all $x \in \mathbb{R}$ and $\mu \in \mathcal{P}_2(\mathbb{R})$. Here we suppose that $Z = (Z_t)_{t\geq 0}$ is a bilateral Gamma process whose scale parameter is 5 and shape parameter is 1. It is straightforward to verify that these coefficients satisfy Conditions **C1–C7** and **T2–T6**.

In the following, we will implement the tamed-adaptive Euler approximation scheme (3.21)-(3.22) with $N = 500$, $x_0 = 1, \ell = 0.3, p_0 = 8$, and $T = 10$. Since the exact solution of equation (3.85) is unknown, we will derive the rate of convergence of the tamed-adaptive Euler approximation scheme (3.21)-(3.22) in an indirect way as in

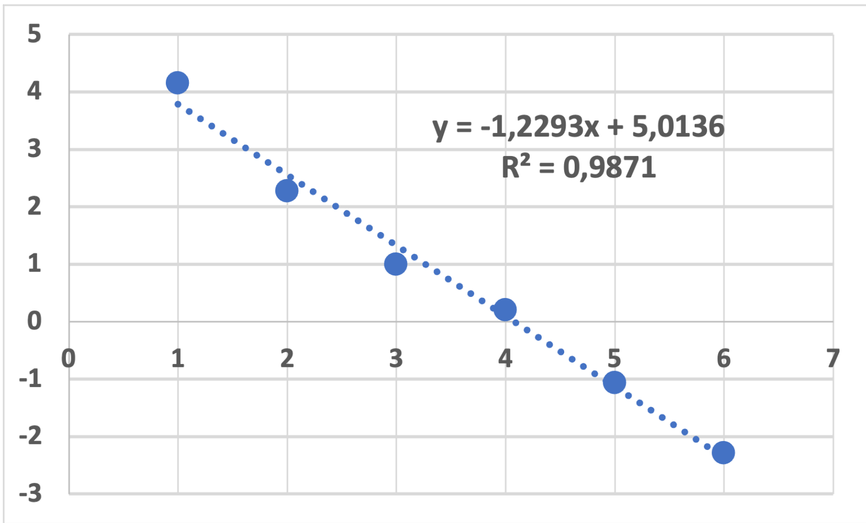

Figure 3.1: Error $\log_2 \mathrm{MSE}(\mathbf{l}, 10)$ plotted against $\mathbf{l} = 1, ..., 6$.

[44, 45]. We consider the mean squared difference of $\widehat{X}$ on two consecutive levels as follows:

$$\mathrm{MSE}(\mathbf{l}, T) = \frac{1}{M} \sum_{k=1}^{M} |\widehat{X}_T^{(\mathbf{l},k)} - \widehat{X}_T^{(\mathbf{l}+1,k)}|^2,$$

where for each $\mathbf{l} \geq 1$, $(\widehat{X}^{(\mathbf{l},k)})_{1\leq k\leq M}$ is a sequence of independent copies of $\widehat{X}^{(\mathbf{l})}$ defined by equations (3.21), (3.22), (3.59) and (3.60) with $\Delta = 2^{-\mathbf{l}}$. Here $\widehat{X}_T^{(\mathbf{l},k)}$ and $\widehat{X}_T^{(\mathbf{l}+1,k)}$ must be simulated to the same Brownian motions and bilateral Gamma processes (See Algorithm 1 in [18]).

It is clear that $\widehat{X}^{(\mathbf{l})}$ converges at some rate of order $\beta \in (0, +\infty)$ in L^2-norm iff $2^{\beta\mathbf{l}}|\widehat{X}_T^{(\mathbf{l}+1)} - \widehat{X}_T^{(\mathbf{l})}|_{L^2} = O(1)$, which implies that $\log_2 \mathrm{MSE}(\mathbf{l}, T) = -2\beta\mathbf{l} + C + o(1)$, for some constant $C \in \mathbb{R}$. Thus we can use the regression method to estimate the rate β. Figure 3.1 shows the values of $\log_2 MSE(\mathbf{l}, T)$ plotted against $\mathbf{l} \in \{1, 2, ..., 6\}$. We see that $\beta \approx 0.5$.

3.9 Conclusions

This chapter extends the "tamed-adaptive" framework, previously developed for standard SDEs in [18, 44, 45], to the significantly more complex domain of multi-dimensional McKean-Vlasov SDEs with jumps.

In particular, we proposed and analyzed a novel tamed-adaptive Euler-Maruyama scheme for the Lévy-driven SDEs (3.1). This scheme is specifically designed for a highly challenging class of equations where all three coefficients — drift (b), diffusion (σ), and jump (c) — are non-globally Lipschitz continuous and exhibit super-linear growth.

We established the strong convergence of this scheme, not only on finite time intervals but also over the infinite time horizon $[0;\infty)$ (under suitable stability conditions).

This result on the infinite time horizon marks a key contribution. Previous adaptive schemes [77] or tamed schemes [68] for McKean-Vlasov SDEs with jumps have, to date, only established convergence over fixed, finite time intervals. To the best of our knowledge, the TAEM scheme presented herein is the first approximation method for this class of McKean-Vlasov SDEs (with super-linear growth in b, σ, and c) shown to converge over an infinite time horizon.

CONCLUSIONS AND RECOMMENDATIONS

Conclusions

This thesis leverages modern tools from stochastic analysis, stochastic differential equations, numerical analysis, and Yamada-Watanabe approximation techniques to construct appropriate approximation schemes for several classes of SDEs with irregular coefficients.

The central results focus on the quantitative and qualitative properties of the exact solutions and the approximated solutions (via the Euler-Maruyama scheme) for SDEs with irregular coefficients and for SDEs with jumps.

The thesis proposes a tamed-adaptive Euler-Maruyama approximation scheme and establishes its strong convergence, over both finite and infinite time intervals, for the following three classes of equations:

- **Classical SDEs:** Applied to a class of SDEs where the drift coefficient is locally Lipschitz continuous and the diffusion coefficient is locally Hölder continuous.
- **Lévy-driven SDEs:** Applied to Lévy-driven SDEs where b (drift) is locally Lipschitz continuous, σ (diffusion) is locally Hölder continuous, and c (jump) is Lipschitz continuous.
- **Lévy-driven McKean-Vlasov SDEs:** Applied to Lévy-driven McKean-Vlasov SDEs where b, σ and c are non-globally Lipschitz continuous and exhibit super-linear growth.

Recommendations

Based on the research conducted in this thesis, we identify the following promising directions for future work:

- Developing and analyzing approximation methods that preserve structural properties. This includes preserving the geometrical or asymptotic properties of stochastic differential equations with complex structures, such as systems of non-colliding random points or positivity-preserving systems.
- Investigating the weak convergence of the tamed-adaptive schemes. The current

work has focused on strong convergence; analyzing the weak convergence properties and rates would be a valuable extension.

- Constructing higher-order approximation schemes. This involves developing schemes with faster convergence rates for SDEs that have smooth coefficients but still exhibit super-linear growth.

LIST OF AUTHOR'S RELATED PAPERS

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