VIETNAM NATIONAL UNIVERSITY, HANOI
VNU UNIVERSITY OF SCIENCE

KIEU TRUNG THUY

THE TAMED-ADAPTIVE EULER-MARUYAMA SCHEME FOR SOME CLASSES OF STOCHASTIC DIFFERENTIAL EQUATIONS WITH IRREGULAR COEFFICIENTS

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ACADEMIC SUPERVISORS:

1. Assoc. Prof. Dr. Ngo Hoang Long
2. Prof. Dr. Nguyen Huu Du

Counter-argument: Prof. D.Sc. Doan Thai Son
Institute of Mathematics, Vietnam Academy of Science and Technology

Counter-argument: Assoc. Prof. Dr. Duong Anh Tuan
Hanoi University of Science and Technology

Counter-argument: Assoc. Prof. Dr. Nguyen Thanh Dieu
Vinh University

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INTRODUCTION

1. Background and Motivation

The Japanese mathematician Gisiro Maruyama is credited as the first to formally and generally propose the pioneering extension of the Euler method from ordinary differential equations to stochastic differential equations. In [64], he introduced the idea of approximating the solution of a stochastic differential equation by a sequence of simpler processes. This work led to the development of one of the most widely used approximation schemes today: the Euler-Maruyama method. Its theoretical importance was later solidified by Kloeden and Platen [46], who proved that this method has a strong convergence order of $\frac{1}{2}$ in the L^2-norm, provided that the coefficients of the equation satisfy the global Lipschitz condition.

This global Lipschitz condition is a critical assumption, and it fails to hold in many important classes of equations. When it is not satisfied, research has focused on several important classes of SDEs with "irregular coefficients", such as when the coefficients are **non-smooth** (e.g., merely Hölder continuous), exhibit **superlinear growth**, or are **singular**.

The challenge posed by superlinearly growing coefficients is particularly severe. For the second class of problems, involving superlinear coefficients, Hutzenthaler, Jentzen, and Kloeden [34] studied the stochastic differential equation

$$dX_t = \left(-\frac{1}{2}\bar{\sigma}^2 X_t + X_t^3\right) dt + \bar{\sigma} X_t dW_t, \qquad \text{with} \qquad X_0 = 1,$$

where the drift coefficient grows superlinearly. They demonstrated that while the moments of the true solution remain finite, the moments of the Euler-Maruyama approximation diverge to infinity. This divergence demonstrates that the Euler-Maruyama scheme fails to converge in the L^p-sense for any $p \geq 1$, highlighting the need for modified numerical schemes.

To overcome this divergence issue, a variety of explicit "tamed" methods have been proposed. Notable examples include the tamed Euler-Maruyama scheme, introduced by Hutzenthaler, Jentzen, and Kloeden [35] and the truncated Euler-Maruyama scheme, proposed by Mao [62]. The fundamental idea behind these "tamed" methods is to replace the original, unbounded coefficients with suitably constructed bounded functions in the numerical scheme. However, these works introduce their own limitation: they are primarily applicable to SDEs whose diffusion coefficients are (at least) locally Lipschitz.

A separate challenge arises from non-smooth coefficients. Hairer, Hutzenthaler, and Jentzen [26] constructed a notable example of a stochastic differential equation where the EM approximation fails to converge even for a bounded and infinitely differentiable (C^∞) drift because its derivatives exhibit extremely high oscillations. For the Cox-Ingersoll-Ross process, where the diffusion coefficient is merely Hölder continuous of order $\frac{1}{2}$, Hefter and Jentzen [27] showed that the convergence rate of any discrete-time approximation can deteriorate significantly. While Gyöngy and Rásonyi [25] later proved the EM scheme can achieve convergence for Hölder continuous diffusion, their results are only applicable to equations whose coefficients have at most linear growth.

This reveals a critical research gap: while "tamed" methods address superlinear growth, they struggle with non-Lipschitz diffusion, and while methods exist for non-Lipschitz diffusion, they often fail for superlinear growth.

A distinct, yet equally significant, limitation of the aforementioned works is their focus on the convergence of numerical schemes over a finite time horizon $[0;T]$. However, the long-term behavior of solutions is of paramount importance in fields such as control theory and optimization. This has motivated research into numerical approximation over infinite time intervals. For instance, Fang and Giles [18] introduced an "adaptive" Euler-Maruyama scheme and proved its convergence in the L^p-norm over an infinite time horizon. Crucially, their analysis still required the diffusion coefficient to be globally Lipschitz continuous.

This focus on long-term behavior is inextricably linked to the fundamental problem of stability. The long-term stability of a system — for instance, the survival or extinction of a species in mathematical biology, is a paramount property that a numerical method must replicate. Foundational results can be found in classical texts such as Khasminskii [42] and Mao [61]. A key challenge is that standard explicit methods like the Euler-Maruyama or Milstein schemes often fail to preserve the stability of the true solution. This shortcoming has spurred the development of alternative methods, such as implicit θ-Euler-Maruyama schemes and various tamed Euler methods [28, 32, 63, 83, 90]. Nevertheless, research on stability-preserving schemes for equations with non-regular coefficients — for example, where the drift or diffusion is merely locally Hölder continuous, remains comparatively limited.

These compelling reasons, which highlight a clear convergence of unsolved challenges, motivate the research topic of this thesis: **"The tamed-adaptive Euler-Maruyama scheme for some classes of stochastic differential equations with irregular coefficients"**.

2. Research objectives and tasks

The primary objectives of this thesis are:

- To establish existence and uniqueness theorems for solutions to certain classes of stochastic differential equations, including those with jumps.
- To propose and analyze novel numerical approximation schemes for these equations, particularly under conditions where the drift coefficient has superlinear growth and is only locally Lipschitz continuous and the diffusion coefficient is merely Hölder continuous (or locally Hölder continuous).
- To investigate the long-term moment stability of both the true solution and its numerical approximation for stochastic differential equations whose drift coefficient satisfies a one-sided Lipschitz condition with a negative constant.

3. Research subjects

The research subjects of the thesis are classes of stochastic differential equations of the forms

$$dX_t = b(X_t)dt + \sigma(X_t)dW_t, \ \ X_0 = x_0;$$

and

$$dX_t = b(X_t)dt + \sigma(X_t)dW_t + c\,(X_{t-})\,dZ_t, \ \ X_0 = x_0.$$

where the drift coefficients $b(x)$ and diffusion coefficients $\sigma(x)$ satisfy one of the following conditions:

- $b(x)$ is locally Lipschitz and one-sided Lipschitz continuous; $\sigma(x)$ is locally $(1/2+\alpha)$-Hölder continuous.
- $b(x)$ is locally Lipschitz and one-sided Lipschitz continuous; $\sigma(x)$ is locally $(1/2+\alpha)$-Hölder continuous and $c(x)$ is Lipschitz continuous.
- $b(x)$, $\sigma(x)$ and c are non-globally Lipschitz continuous and of superlinear growth.

4. Research scope

The research scope of this thesis encompasses stochastic analysis, stochastic differential equations, and numerical analysis. The primary contributions of this work focus on the quantitative and qualitative properties of both exact and approximate solutions, specifically concerning the Euler-Maruyama scheme for stochastic differential equations with irregular coefficients and for stochastic differential equations with jumps.

5. Research methods

- Conduct a thorough analysis of contemporary research on stochastic differential equations with irregular coefficients and associated approximation methods.
- Execute computer simulations to analyze, evaluate, and propose innovative approximation algorithms.
- Engage in scientific exchange initiatives, including conferences and seminars, to facilitate the exchange, discussion, and updating of cutting-edge research methodologies and findings within the relevant fields.

6. Contributions of the thesis

The results of this thesis contribute to the expansion of research in the numerical solutions of specific classes of stochastic differential equations. It is anticipated that this thesis will yield several novel contributions:

- Propose a numerical scheme that achieves strong convergence over both finite and infinite time intervals for a class of one-dimensional stochastic differential equations with locally Lipschitz continuous drift and locally Hölder continuous diffusion coefficients.
- Develop a tamed-adaptive Euler-Maruyama approximation scheme for Lévy-driven stochastic differential equations where coefficient σ is locally Hölder continuous, coefficients σ and b exhibit superlinear growth, and coefficient c is Lipschitz continuous.
- Introduce a tamed-adaptive Euler-Maruyama approximation scheme for Lévy-driven McKean-Vlasov stochastic differential equations where coefficients σ and b are non-globally Lipschitz continuous, superlinearly growing, and coefficient c is Lipschitz continuous.

The thesis can be used for reference in related research by students and scientists in the fields of probability theory and mathematical statistics and in the field of numerical analysis.

7. Structure of the thesis

The thesis includes an introduction, three main chapters, conclusions, list of published works and references:

- **Chapter 1:** Overview
- **Chapter 2:** Tamed-adaptive Euler-Maruyama scheme for Lévy-driven SDEs with irregular coefficients

- **Chapter 3:** Tamed-adaptive Euler-Maruyama scheme for Lévy-driven McKean-Vlasov SDEs with irregular coefficients

The thesis is written based on 03 published articles.

Chương 1

OVERVIEW

In this chapter, we review the foundational concepts and summarize the key previous results that form the basis for our research.

1.1 Stochastic differential equation driven by Brownian motion

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions. Let $W_t = \left(W_t^1, W_t^2 \ldots, W_t^m\right)^\top$, $t \geq 0$ be an m-dimensional Brownian motion defined on the space. Let $0 \leq t_0 < T < \infty$ and X_0 be an $\mathcal{F}_{t_0}$-measurable $\mathbb{R}^d$-valued random variable such that $\mathbb{E}\left[\|x_0\|^2\right] < \infty$. Consider the d-dimensional stochastic differential equation of Itô type

$$dX_t = b(t, X_t)dt + \sigma(t, X_t)dW_t \quad \text{on } t_0 \leq t \leq T \tag{1.1}$$

with initial value $X(t_0) = X_0$. By the definition of stochastic differential, this equation is equivalent to the following stochastic integral equation:

$$X(t) = x_0 + \int_{t_0}^t b(s, X(s))ds + \int_{t_0}^t \sigma(s, X(s))dW(s) \quad \text{on } t_0 \leq t \leq T. \tag{1.2}$$

Theorem 1.1.11 *Assume that there exist two positive constants K and $\overline{K}$ such that*

(i) *(Lipschitz condition)) for all $x, y \in \mathbb{R}^d$ và $t \in [t_0, T]$*

$$\|b(t,x) - b(t,y)\|^2 \vee \|\sigma(t,x) - \sigma(t,y)\|^2 \leq \overline{K}\|x-y\|^2. \tag{1.3}$$

(ii) *(Linear growth condition)) for all $x, y \in \mathbb{R}^d \times [t_0, T]$*

$$\|b(t,x)\|^2 \vee |\sigma(t,x)|^2 \leq K(1 + \|x\|^2). \tag{1.4}$$

Then there exists a unique solution $X(t)$ to equation (1.1) and the solution such that

$$\mathbb{E}\left[\int_{t_0}^T \|X_s\|^2 ds\right] < \infty. \tag{1.5}$$

Theorem 1.1.14. *Let $p \geq 2$ and $x_0 \in L^p\left(\Omega; \mathbb{R}^d\right)$. Assume that there exists a constant $\alpha > 0$ such that for all $(t,x) \in [t_0,T] \times \mathbb{R}^d$,*

$$x^\top b(t,x) + \frac{p-1}{2}\|\sigma(t,x)\|^2 \leq \alpha\left(1+\|x\|^2\right). \tag{1.6}$$

Then

$$\mathbb{E}\left[|X_t|^p\right] \leq 2^{\frac{p-2}{2}}\left(1+\mathbb{E}\left[|x_0|^p\right]\right)e^{p\alpha(t-t_0)} \quad \text{for all } t \in [t_0,T]. \tag{1.7}$$

1.2 Stochastic differential equations driven by Lévy process

Proposition 1.2.23. (**Lévy-Itô decomposition**) *Let $Z = (Z_t)_{t\geq 0}$ be a Lévy process on $\mathbb{R}^d$ and ν be its Lévy measure.*

- *ν is a Radon measure on $\mathbb{R}_0^d$ and verifies:*

$$\int_{\mathbb{R}_0^d}(1\wedge|z|^2)\nu(dz) < \infty.$$

- *The jump measure of Z, denoted by N, is a Poisson random measure on $[0,\infty]\times\mathbb{R}^d$ with intensity measure $\nu(dz)dt$.*
- *There exist a vector γ and a d-dimensional Brownian motion $(W_t)_{t\geq 0}$ with covariance matrix A such that*

$$Z_t = \gamma t + W_t + \int_0^t\int_{|z|\geq 1} zN(ds,dz) + \int_0^t\int_{|z|<1} z\widetilde{N}(ds,dz). \tag{1.8}$$

The terms in (1.8) are independent.

Note that, in the case that $Z = (Z_t)_{t\geq 0}$ is a d-dimensional centered pure jump Lévy process whose Lévy measure ν satisfies $\int_{\mathbb{R}^d}\left(1\wedge|z|^2\right)\nu(dz) < +\infty$, the Lévy-Itô decomposition of Z is given by

$$Z_t = \int_0^t\int_{\mathbb{R}_0^d} z(N(ds,dz)-\nu(dz)ds).$$

Proposition 1.2.24. (**Lévy-Khinchin representation**) *Let $Z = (Z_t)_{t\geq 0}$ be a Lévy process on $\mathbb{R}^d$. There exists a continuous function $\psi : \mathbb{R}^d \mapsto \mathbb{R}$ called the characteristic exponent of Z, such that:*

$$\mathbb{E}\left[e^{iuZ_t}\right] = e^{t\psi(u)}, \quad u \in \mathbb{R}^d. \tag{1.9}$$

where

$$\psi(u) = i\gamma u - \frac{1}{2}u\cdot Au + \int_{\mathbb{R}^d}\left(e^{iuz}-1-iuz\mathbf{1}_{\{|z|\leq 1\}}\right)\nu(dz)$$

The triplet (A, ν, γ) is called *characteristic triplet* or *Lévy triplet* of the process Z_t.

Next, we now state the Burkholder-Davis-Gundy inequality for the compensated Poisson stochastic integral which will be useful in the thesis.

Lemma 1.2.25. *Let $\mathcal{B}(\mathbb{R}_0^d)$ be the Borel σ-algebra of $\mathbb{R}_0^d$ and $\mathcal{P}$ be the progressive σ-algebra on $\mathbb{R}_+ \times \Omega$. Let g be a $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}_0^d)$-measurable function satisfying that $\int_0^T \int_{\mathbb{R}_0^d} |g(s,z)|^2 \nu(dz)ds < \infty$ $\mathbb{P}$-a.s. for all $T \geq 0$. Then for any $p \geq 2$, there exists a positive constant C_p such that*

$$\mathbb{E}\left[\sup_{t\in[0,T]}\left|\int_0^t\int_{\mathbb{R}_0^d} g(s,z)\widetilde{N}(ds,dz)\right|^p\right]$$
$$\leq C_p\left(\mathbb{E}\left[\left(\int_0^T\int_{\mathbb{R}_0^d}|g(s,z)|^2\nu(dz)ds\right)^{\frac{p}{2}}\right]+\mathbb{E}\left[\int_0^T\int_{\mathbb{R}_0^d}|g(s,z)|^p\nu(dz)ds\right]\right).$$

Furthermore, for any $1 \leq p < 2$, there exists a positive constant C_p such that

$$\mathbb{E}\left[\sup_{t\in[0,T]}\left|\int_0^t\int_{\mathbb{R}_0^d} g(s,z)\widetilde{N}(ds,dz)\right|^p\right] \leq C_p\mathbb{E}\left[\left(\int_0^T\int_{\mathbb{R}_0^d}|g(s,z)|^2\nu(dz)ds\right)^{\frac{p}{2}}\right].$$

Next, consider processes $X = (X_t)_{t\geq 0}$, admitting stochastic integral representation in the form

$$X_t = x + \int_0^t b(X_s)ds + \int_0^t \sigma(X_s)dW_s + \int_0^t\int_{\mathbb{R}_0} c(X_{s^-},z)\widetilde{N}(ds,dz). \tag{1.10}$$

Theorem 1.2.26. The one-dimensional Itô formula. *Let $X = (X_t)_{t\geq 0}$, be the Itô-Lévy process given by (1.10) and let $f : (0,\infty)\times\mathbb{R} \to \mathbb{R}$ be a function in $C^{1,2}((0,\infty)\times\mathbb{R})$ and define*

$$Y_t := f(t, X_t), \quad t \geq 0.$$

Then the process $Y = (Y_t)_{t\geq 0}$, is also an Itô-Lévy process and its differential form is given by

$$\begin{aligned}
dY_t = {} & \frac{\partial f}{\partial t}(t,X_t)dt + \frac{\partial f}{\partial x}(t,X_t)b(X_t)dt + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(t,X_t)\sigma^2(X_t)dt \\
& + \frac{\partial f}{\partial x}(t,X_t)\sigma(X_t)dW_t \\
& + \int_{\mathbb{R}_0}\left[f(t,X_t+c(X_t,z)) - f(t,X_t) - \frac{\partial f}{\partial x}(t,X_t)c(X_t,z)\right]\nu(dz)dt \\
& + \int_{\mathbb{R}_0}\left[f\left(t,X_{t^-}+c(X_{t^-},z)\right) - f\left(t,X_{t^-}\right)\right]\widetilde{N}(dt,dz).
\end{aligned} \tag{1.11}$$

Theorem 1.2.27. The multi-dimensional Itô formula. *Let $X = (X_t)_{t\geq 0}$, be an d-dimensional Itô-Lévy process. Let $f : (0,\infty)\times \mathbb{R}^d \to \mathbb{R}$ be a function in $C^{1,2}\left((0,\infty)\times\mathbb{R}^d\right)$ and define*

$$Y_t := f(t, X_t), \quad t \geq 0.$$

Then the process $Y = (Y_t)_{t\geq 0}$, is a one-dimensional Itô-Lévy process and its differential form is given by

$$
\begin{aligned}
& dY(t) \\
&= \frac{\partial f}{\partial t}(t, X_t)dt + \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(t, X_t) b_i(X_t)dt + \sum_{i=1}^{d}\sum_{j=1}^{d} \frac{\partial f}{\partial x_i}(t, X_t)\sigma_{ij}(X_t)dW_t^j \\
&+ \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{d} \frac{\partial^2 f}{\partial x_i \partial x_j}(t, X_t)\left(\sigma\sigma^T\right)_{ij}(X_t)dt + \sum_{j=1}^{d}\int_{\mathbb{R}_0}\left[f\left(t, X_t + c^{(j)}(X_t, z)\right)\right. \\
&\left. - f(t, X_t) - \sum_{i=1}^{d}\frac{\partial f}{\partial x_i}(t, X_t)c_{ij}(X_t, z)\right]\nu_j\left(dz_j\right)dt \\
&+ \sum_{j=1}^{d}\int_{\mathbb{R}_0}\left[f\left(t, X_{t^-} + c^{(j)}(X_{t^-}, z)\right) - f\left(t, X_{t^-}\right)\right]\widetilde{N}_j\left(dt, dz_j\right),
\end{aligned}
\tag{1.12}
$$

where $c^{(j)}$ is the column number j of the $d \times d$ matrix $c = [c_{ij}]$.

Finally, we consider a process $X = (X_t)_{t\geq 0}$, the solution to the following stochastic differential equation with jumps in $\mathbb{R}^d$:

$$
dX_t = b\left(X_t\right)dt + \sum_{j=1}^{d}\sigma_j\left(X_t\right)dW_t^j + \int_{\mathbb{R}_0} c\left(X_{t-}, z\right)\widetilde{N}(dt, dz),
$$

with initial condition $X_0 = x_0 \in \mathbb{R}^d$. The coefficients $\sigma_j, b : \mathbb{R}^d \to \mathbb{R}^d$ and $c : \mathbb{R}^d \times \mathbb{R}_0 \to \mathbb{R}^d$ are measurable functions satisfying the following Lipschitz and linear growth conditions: for all $x, y \in \mathbb{R}^d$,

$$
\max_j\left\{\left|\sigma_j(x) - \sigma_j(y)\right|^2; |b(x) - b(y)|^2; \int_{\mathbb{R}_0}|c(x,z) - c(y,z)|^2\nu(dz)\right\} \leq K|x-y|^2
$$

and

$$
\max_j\left\{\left|\sigma_j(x)\right|^2; |b(x)|^2; \int_{\mathbb{R}_0}|c(x,z)|^2\nu(dz)\right\} \leq K\left(1 + |x|^2\right).
$$

Theorem 1.2.28. *There exists a unique càdlàg, adapted, and Markov process X on $(\Omega, \mathcal{F}, \mathbb{P})$ satisfying the integral equation*

$$
X_t = x_0 + \int_0^t b\left(X_s\right)ds + \sum_{j=1}^{d}\int_0^t \sigma_j\left(X_s\right)dW_s^j + \int_0^t\int_{\mathbb{R}_0} c\left(X_{s-}, z\right)\widetilde{N}(ds, dz).
$$

Moreover, for any $T > 0$,

$$
\mathbb{E}\left[\sup_{t\in[0,T]}|X_t|^2\right] < \infty.
$$

Chương 2

TAMED-ADAPTIVE EULER-MARUYAMA SCHEME FOR LÉVY-DRIVEN SDEs WITH IRREGULAR COEFFICIENTS

This chapter introduces a novel approximation scheme for a class of stochastic differential equations (SDEs) with irregular coefficients. We address both SDEs driven by Brownian motion (continuous noise) and the more general case of SDEs driven by Lévy processes (which include jumps). Since the Brownian-driven model is a special case of the Lévy model and the proof methods are logically parallel, this chapter, for the sake of conciseness, presents the detailed analysis only for the more general Lévy case. The corresponding results for the simpler Brownian-driven model can be inferred similarly. The results presented herein are based on the author's publications [1, 2] listed in the **List of Author's Related Papers** section.

2.1 Model assumptions

We consider the process $X = (X_t)_{t\geq 0}$ as a solution to the following stochastic differential equation (SDE) with jumps

$$X_t = x_0 + \int_0^t b(X_s)ds + \int_0^t \sigma(X_s)dW_s + \int_0^t c\left(X_{s-}\right)dZ_s. \tag{2.1}$$

The integral equation of (2.1) can be written as

$$X_t = x_0 + \int_0^t b(X_s)ds + \int_0^t \sigma(X_s)dW_s + \int_0^t \int_{\mathbb{R}_0} c\left(X_{s-}\right) z\widetilde{N}(ds, dz).$$

Assume that coefficients b, σ, c and the Lévy measure ν satisfy the following conditions:

B1. There exists a positive constant L_0 such that

$$|c(x)| \leq L_0(1 + |x|), \quad \forall x \in \mathbb{R}.$$

B2. For some $p_0 \in [2; +\infty)$, there exist constants $\gamma \in \mathbb{R}$, $\eta \in [0; +\infty)$ such that

$$xb(x) + \frac{p_0 - 1}{2}\sigma^2(x) + \frac{c^2(x)}{2L_0}\int_{\mathbb{R}_0} |z|\left((1 + L_0|z|)^{p_0-1} - 1\right)\nu(dz) \leq \gamma x^2 + \eta, \quad \forall x \in \mathbb{R}.$$

B3. Coefficient b is locally Lipschitz: for any $R > 0$, there exists a positive constant L_R such that

$$|b(x) - b(y)| \le L_R|x - y|, \quad \forall |x| \vee |y| \le R.$$

B4. Coefficient σ is locally $\left(\alpha + \frac{1}{2}\right)$ -Hölder continuous: for any $R > 0$, there exist positive constants L_R and $\alpha \in \left(0; \frac{1}{2}\right]$ such that

$$|\sigma(x) - \sigma(y)| \le L_R|x - y|^{1/2+\alpha}, \quad \forall |x| \vee |y| \le R.$$

B5. Coefficient c is locally Lipschitz: for any $R > 0$, there exists a positive constant L_R such that

$$|c(x) - c(y)| \le L_R|x - y|, \quad \forall |x| \vee |y| \le R.$$

B6. $\int_{|z|>1} |z|^p \nu(dz) < \infty$ for all $p \in [1; 2p_0]$ and $\int_{0<|z|\le 1} |z|\nu(dz) < \infty$.

B7. Coefficient b is one-sided Lipschitz: there exists a constant L_1 such that

$$(x - y)(b(x) - b(y)) \le L_1|x - y|^2, \quad \forall x, y \in \mathbb{R}.$$

B8. Coefficient b is locally Lipschitz continuous: there exist positive constants l and L_2 such that

$$|b(x) - b(y)| \le L_2 \left(1 + |x|^l + |y|^l\right) |x - y|, \quad \forall x, y \in \mathbb{R}.$$

B9. Coefficient σ is $(\alpha + \frac{1}{2})$-locally Hölder continuous: there exist positive constants m, L_3 and $\alpha \in [0; \frac{1}{2}]$ such that

$$|\sigma(x) - \sigma(y)| \le L_3 \left(1 + |x|^m + |y|^m\right) |x - y|^{1/2+\alpha}, \quad \forall x, y \in \mathbb{R}.$$

B10. Coefficient c is Lipschitz: there exists a positive constant L_4 such that

$$|c(x) - c(y)| \le L_4|x - y|, \quad \forall x, y \in \mathbb{R}.$$

2.2 Lévy-driven SDEs with irregular coefficients

Proposition 2.3.1. *Assume that coefficients b, c, σ and the Lévy measure ν satisfy conditions **B1, B2, B6** and σ is bounded on every compact subset of $\mathbb{R}$. Assume further that $X = (X_t)_{t\ge 0}$ is a solution to equation (2.1). Then, for any $p \in (0, p_0]$, there exists a positive constant C_p such that for any $t \ge 0$,*

$$\mathbb{E}\left[|X_t|^p\right] \le \begin{cases} C_p(1 + e^{\gamma p t}) & \text{if } \gamma \ne 0, \\ C_p(1 + t)^{p/2} & \text{if } \gamma = 0. \end{cases} \tag{2.2}$$

Note that when $\gamma < 0$, we have $\sup_{t\ge 0} \mathbb{E}\left[|X_t|^p\right] \le 2C_p$.

Theorem 2.3.3. *Assume that the coefficients b, c and σ satisfy the conditions **B1**–**B5**. Assume further that the Lévy measure satisfies $\int_{\mathbb{R}_0} |z|\nu(dz) < \infty$ and $\int_{\mathbb{R}_0} z^2\nu(dz) < \infty$. Then the path-wise uniqueness holds for equation (2.1).*
Moreover, suppose that there exist positive constants C and $\ell \in (0; \frac{p_0}{4}]$ such that

$$|b(x)| \vee |\sigma(x)| \vee |c(x)| \leq C\left(1 + |x|^\ell\right),$$

*for all $x \in \mathbb{R}$, where p_0 is defined in Condition **B2**. Then the equation (2.1) has a strong solution.*

2.3 Tamed-adaptive Euler-Maruyama scheme

For each $\Delta \in (0, 1)$, the tamed-adaptive Euler-Maruyama discretisation of equation (2.1) is defined as follows

$$\begin{cases} t_0 = 0, \quad \widehat{X}_0 = x_0, \quad t_{k+1} = t_k + h(\widehat{X}_{t_k})\Delta, \\ \widehat{X}_{t_{k+1}} = \widehat{X}_{t_k} + b(\widehat{X}_{t_k})\,(t_{k+1} - t_k) + \sigma_\Delta(\widehat{X}_{t_k})(W_{t_{k+1}} - W_{t_k}) \\ \qquad\qquad + c_\Delta\left(\widehat{X}_{t_k}\right)\left(Z_{t_{k+1}} - Z_{t_k}\right), \end{cases} \tag{2.3}$$

where

$$h(x) = \frac{1}{(1 + |b(x)| + |\sigma(x)| + |x|^l)^2 + |c(x)|^{p_0}}, \tag{2.4}$$

The next result provides a sufficient condition for $t_k \to \infty$ as $k \to \infty$, which implies that the tamed adaptive approximation scheme (2.3) is well-defined.

Proposition 2.4.1. *Suppose that there exist positive constants L and β such that the coefficients $b, c, \sigma, c_\Delta, \sigma_\Delta$ satisfy the following conditions*

T1. $|b(x)| \vee |\sigma(x)| \leq L\left(1 + |x|^\beta\right)$;

T2. $x(b(x) - b(0)) \leq L|x|^2$;

T3. $|\sigma_\Delta(x)| \leq L|\sigma(x)|$ *and* $|c_\Delta(x)| \leq |c(x)|$;

T4. $|\sigma_\Delta(x)| \leq \frac{L}{\sqrt{\Delta}}$; $|c_\Delta(x)| \leq \frac{L}{\sqrt{\Delta}}$ *and* $|b(x)c_\Delta(x)| \leq \frac{L}{\sqrt{\Delta}}$;

for any $x \in \mathbb{R}$. Assume further that the Lévy measure satisfies $\int_{\mathbb{R}_0} z^2\nu(dz) < \infty$. Then

$$\lim_{k \to +\infty} t_k = +\infty \quad a.s.$$

2.4 Moments of the approximate solution

Theorem 2.5.2. *Assume that Conditions **T1–T4** and **B6** hold, and for some* $p_0 \in [2;+\infty)$, *there exist constants* $\gamma \in \mathbb{R}$, $\eta \in [0;+\infty)$ *such that for all* $x \in \mathbb{R}$,

$$xb(x) + \frac{p_0 - 1}{2}\sigma_\Delta^2(x) + \frac{c_\Delta^2(x)}{2L_0}\int_{\mathbb{R}_0} |z|\left((1 + L_0|z|)^{p_0-1} - 1\right)\nu(dz) \leq \gamma x^2 + \eta. \tag{2.5}$$

Then, for any positive integer $k \leq p_0/2$, *there exists a positive constant* $C = C(x_0, k, \eta, \gamma, L, L_0, p_0)$ *which does not depend neither on* t *nor on* Δ *such that*

$$\mathbb{E}\left[|\widehat{X}_t|^{2k}\right] \vee \mathbb{E}\left[|\widehat{X}_{\underline{t}}|^{2k}\right] \leq \begin{cases} Ce^{2k\gamma t} & \textit{if } \gamma > 0, \\ C(1+t)^k & \textit{if } \gamma = 0, \\ C & \textit{if } \gamma < 0. \end{cases} \tag{2.6}$$

2.5 Convergence of the tamed-adaptive Euler-Maruyama scheme

Lemma 2.6.1. *Suppose that coefficients* $b, c, \sigma, \sigma_\Delta, c_\Delta$ *and the Lévy measure* ν *satisfy all conditions of Theorem 2.5.2 and* $p \in (0;p_0]$, *then there exists a positive constant* $C_p = C(p, L)$ *such that*

$$\sup_{t\geq 0}\mathbb{E}\left[|\widehat{X}_t - \widehat{X}_{\underline{t}}|^p\right] \leq C_p\Delta^{1\wedge p/2},$$

Theorem 2.6.2. *Assume that Conditions **B2, B6–B10** hold and* $p_0 \geq \max\{4l; 2 + 4\alpha + 4m\}$. *Assume that the functions* $c, b, \sigma, c_\Delta, \sigma_\Delta$ *and the Lévy measure* ν *satisfy all conditions of Theorem 2.5.2, and*

$$|c(x) - c_\Delta(x)| \leq L_5\Delta^{1/2}c^2(x)(1 + |b(x)|), \quad |\sigma(x) - \sigma_\Delta(x)| \leq L_5\Delta^{1/2}\sigma^2(x), \tag{2.7}$$

for all $x \in \mathbb{R}$ *and some constant* $L_5 > 0$.
Then, for any $T > 0$, *there exists a positive constant* $C_T = C(x_0, L, L_0, L_1, L_2, L_3, L_4, L_5, \gamma, \eta, T)$ *such that*

$$\sup_{0\leq t\leq T}\mathbb{E}\left[|\widehat{X}_t - X_t|\right] \leq \begin{cases} C_T\Delta^\alpha & \textit{if } 0 < \alpha \leq \frac{1}{2}, \\ \frac{C_T}{\log\frac{1}{\Delta}} & \textit{if } \alpha = 0. \end{cases} \tag{2.8}$$

Moreover, let $\mu := \int_{\mathbb{R}_0} |z|\nu(dz)$ *and assume that* $L_1 + 2L_4\mu < 0$, $\gamma < 0$, *then there exists a positive constant* $C = C(x_0, L, L_0, L_1, L_2, L_3, L_4, L_5, \gamma, \eta)$ *which does not depend on* T *such that*

$$\sup_{t\geq 0}\mathbb{E}\left[|\widehat{X}_t - X_t|\right] \leq \begin{cases} C\Delta^\alpha & \textit{if } 0 < \alpha \leq \frac{1}{2}, \\ \frac{C}{\log\frac{1}{\Delta}} & \textit{if } \alpha = 0. \end{cases} \tag{2.9}$$

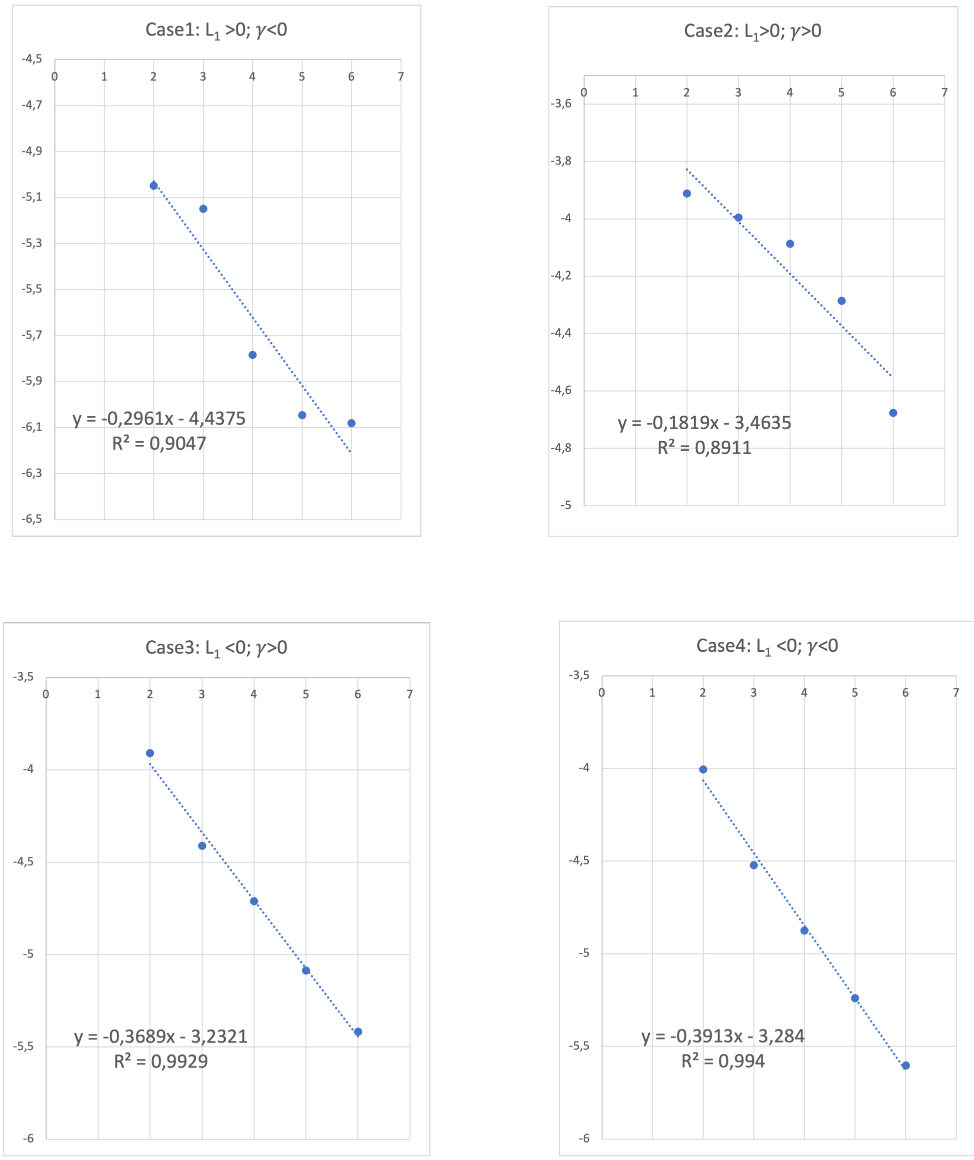
2.6 Numerical experiments

Consider numerical experiments for four SDEs with coefficients given in Table 2.1.

Case	b	σ	c	p_0	L_1	γ	η	l	m	α
1	$-1+x-x^3$	$1+(1+x)x^{2/3}$	$x+\sin(x)$	10	1	-1	31873	2	$\frac{4}{3}$	$\frac{1}{6}$
2	$-1+x-x^3$	$1+\sqrt{\frac{x^4+x^{4/3}}{14}}$	$x+\sin(x)$	10	1	1	957	2	2	$\frac{1}{6}$
3	$-1-x-x^{7/3}$	$1+\sqrt{\frac{2x^2+x^{10/3}+x^{4/3}}{14}}$	$x+\sin(x)$	10	-1	1	1868	$\frac{4}{3}$	1	$\frac{1}{6}$
4	$-1-x-x^{7/3}$	$1+\sqrt{\frac{x^{10/3}+x^{4/3}}{14}}$	$x+\sin(x)$	10	-1	-1	1583	$\frac{4}{3}$	1	$\frac{1}{6}$

Bảng 2.1: Four jump SDEs with their parameters.

Figure 2.1 shows the simulation result of $\log_2 me(l)$ for $l = 2, ..., 6$. We draw the regression lines to estimate the empirical rates of convergence β in each case.



Hình 2.1: Values of $\log_2(me(l))$ for $l = 2, 3, 4, 5, 6$.

Chương 3

TAMED-ADAPTIVE EULER-MARUYAMA SCHEME FOR LÉVY-DRIVEN MCKEAN-VLASOV SDEs WITH IRREGULAR COEFFICIENTS

Following the analysis of the tamed-adaptive Euler-Maruyama scheme for stochastic differential equations with jumps in Chapter 2, this chapter extends the investigation to the numerical approximation of a more complex class of equations, namely McKean-Vlasov SDEs with jumps. The fundamental challenge of these equations is that their coefficients depend on both the state and the probability distribution of the process, creating a complex mean-field interaction structure. In this chapter, we focus on the case where the drift and diffusion coefficients are non-globally Lipschitz continuous and exhibit superlinear growth. The results presented herein are based on the author's publication [3], listed in the **List of Author's Related Papers** section.

3.1 Model assumptions

On a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, we consider the d-dimensional process $X = (X_t)_{t\geq 0}$ is the solution to the following McKean-Vlasov stochastic differential equation (SDE) with jumps

$$dX_t = b(X_t, \mathcal{L}_{X_t})dt + \sigma(X_t, \mathcal{L}_{X_t})dW_t + c\left(X_{t-}, \mathcal{L}_{X_{t-}}\right)dZ_t, \tag{3.1}$$

for $t \geq 0$, where $X_0 = x_0 \in \mathbb{R}^d$ is a fixed initial value, $\mathcal{L}_{X_t}$ denotes the marginal law of the process X at time t, $W = (W_t)_{t\geq 0}$ is a d-dimensional standard Brownian motion and $Z = (Z_t)_{t\geq 0}$ is a d-dimensional centered pure jump Lévy process whose Lévy measure ν satisfies $\int_{\mathbb{R}^d}(1 \wedge |z|^2)\nu(dz) < +\infty$. The processes W and Z are assumed to be independent. The natural filtration $(\mathcal{F}_t)_{t\geq 0}$ is generated by W and Z.

We denote by $\mathcal{P}(\mathbb{R}^d)$ the space of all probability measures defined on a measurable space $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$, where $\mathcal{B}(\mathbb{R}^d)$ denotes the Borel σ-field over $\mathbb{R}^d$, and by

$$\mathcal{P}_2(\mathbb{R}^d) := \left\{\mu \in \mathcal{P}(\mathbb{R}^d) : \int_{\mathbb{R}^d} |x|^2\mu(dx) < \infty\right\}$$

the subset of probability measures with finite second moment. The space $\mathcal{P}_2(\mathbb{R}^d)$ is equipped with the $\mathcal{L}_2$-Wasserstein distance. That is, for $\mu, \nu \in \mathcal{P}_2(\mathbb{R}^d)$, the $\mathcal{L}_2$-Wasserstein distance between μ and ν is defined as

$$\mathcal{W}_2(\mu,\nu) := \inf_{\pi\in\mathcal{C}(\mu,\nu)} \left(\int_{\mathbb{R}^d\times\mathbb{R}^d} |x-y|^2\, \pi(dx,dy) \right)^{1/2},$$

where $\mathcal{C}(\mu,\nu)$ denotes all the couplings of μ and ν, i.e., $\pi \in \mathcal{C}(\mu,\nu)$ if and only if $\pi(\cdot,\mathbb{R}^d) = \mu(\cdot)$ and $\pi(\mathbb{R}^d,\cdot) = \nu(\cdot)$.

The coefficients $b = (b_i)_{1\le i\le d} : \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}^d$, $\sigma = (\sigma_{ij})_{1\le i,j\le d} : \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}^d \otimes \mathbb{R}^d$ and $c = (c_{ij})_{1\le i,j\le d} : \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}^d \otimes \mathbb{R}^d$ are measurable functions. The integral equation (3.1) can be written as

$$X_t = x_0 + \int_0^t b(X_s, \mathcal{L}_{X_s})ds + \int_0^t \sigma(X_s,\mathcal{L}_{X_s})dW_s + \int_0^t \int_{\mathbb{R}_0^d} c\left(X_{s-},\mathcal{L}_{X_{s-}}\right) z\widetilde{N}(ds,dz),$$

for any $t \ge 0$.

Assume that the drift, diffusion and jump coefficients b, σ, c and the Lévy measure ν of equation (3.1) satisfy the following conditions:

C1. There exists a positive constant L such that

$$2\langle x, b(x,\mu)\rangle + |\sigma(x,\mu)|^2 + |c(x,\mu)|^2 \int_{\mathbb{R}_0^d} |z|^2\nu(dz) \le L\left(1+|x|^2+\mathcal{W}_2^2(\mu,\delta_0)\right),$$

for any $x \in \mathbb{R}^d$ and $\mu \in \mathcal{P}_2(\mathbb{R}^d)$.

C2. There exist constants $\kappa_1 > 0, \kappa_2 > 0$, $L_1 \in \mathbb{R}$ and $L_2 \ge 0$ such that

$$\begin{aligned}&2\langle x-\overline{x}, b(x,\mu)-b(\overline{x},\overline{\mu})\rangle + \kappa_1|\sigma(x,\mu)-\sigma(\overline{x},\overline{\mu})|^2\\ &+\kappa_2|c(x,\mu)-c(\overline{x},\overline{\mu})|^2\int_{\mathbb{R}_0^d}|z|^2\nu(dz) \le L_1|x-\overline{x}|^2 + L_2\mathcal{W}_2^2(\mu,\overline{\mu}),\end{aligned}$$

for any $x, \overline{x} \in \mathbb{R}^d$ and $\mu, \overline{\mu} \in \mathcal{P}_2(\mathbb{R}^d)$.

C3. $b(x,\mu)$ is a continuous function of $x \in \mathbb{R}^d$ and $\mu \in \mathcal{P}_2(\mathbb{R}^d)$.

C4. There exist constants $L > 0$ and $\ell \ge 1$ such that

$$|b(x,\mu)-b(\overline{x},\overline{\mu})| \le L\left(1+|x|^\ell+|\overline{x}|^\ell\right)\left(|x-\overline{x}|+\mathcal{W}_2(\mu,\overline{\mu})\right),$$

for any $x, \overline{x} \in \mathbb{R}^d$ and $\mu, \overline{\mu} \in \mathcal{P}_2(\mathbb{R}^d)$.

C5. There exists an even integer $p_0 \in [2;+\infty)$ such that $\int_{|z|>1}|z|^{p_0}\nu(dz) < \infty$ and $\int_{0<|z|\le 1}|z|\nu(dz) < \infty$.

C6. There exists a positive constant L_3 such that

$$|c(x,\mu)| \leq L_3\left(1+|x|+\mathcal{W}_2(\mu,\delta_0)\right),$$

for any $x\in\mathbb{R}^d$ and $\mu\in\mathcal{P}_2(\mathbb{R}^d)$.

C7. For the even integer $p_0\in[2;+\infty)$ given in **C5** and the positive constant L_3 given in **C6**, there exist constants $\gamma_1\in\mathbb{R}$, $\gamma_2\geq 0$ and $\eta\geq 0$ such that

$$\begin{aligned}&\langle x,b(x,\mu)\rangle+\frac{p_0-1}{2}|\sigma(x,\mu)|^2+\frac{1}{2L_3}|c(x,\mu)|^2\int_{\mathbb{R}_0^d}|z|\left((1+L_3|z|)^{p_0-1}-1\right)\nu(dz)\\ \leq&\gamma_1|x|^2+\gamma_2\mathcal{W}_2^2(\mu,\delta_0)+\eta,\end{aligned}$$

for any $x\in\mathbb{R}^d$ and $\mu\in\mathcal{P}_2(\mathbb{R}^d)$.

3.2 Lévy-driven McKean-Vlasov SDEs with irregular coefficients

We begin by recalling a key result on the existence and uniqueness of the strong solution for this SDE, established in [68].

Proposition 3.3.1. *Assume Conditions **C1**, **C3** hold and Condition **C2** holds for* $\kappa_1=\kappa_2=1, L_1=L_2>0$. *Then, there exists a unique càdlàg process* $X=(X_t)_{t\geq 0}$ *taking values in* $\mathbb{R}^d$ *satisfying the McKean-Vlasov SDE with jumps* (3.1) *such that*

$$\sup_{t\in[0,T]}\mathbb{E}\left[|X_t|^2\right]\leq K,$$

where $T>0$ *is a fixed constant and* $K:=K(|x_0|^2,d,L,L_1,T)$ *is a positive constant.*

Next, we demonstrate the following moment estimates for the exact solution $X=(X_t)_{t\geq 0}$ of the McKean-Vlasov stochastic differential equation with jumps (3.1).

Proposition 3.3.2. *Let* $X=(X_t)_{t\geq 0}$ *be a solution to equation* (3.1). *Assume Conditions **C6**, **C7** hold,* σ *is bounded on* $C\times\mathcal{P}_2(\mathbb{R}^d)$ *for every compact subset* C *of* $\mathbb{R}^d$, *and **C5** holds for* $q=2p_0$. *Then for any* $p\in[2;p_0]$, *there exists a positive constant* C_p *such that for any* $t\geq 0$

$$\mathbb{E}\left[|X_t|^p\right]\leq\begin{cases}C_p(1+e^{\gamma pt}) & \text{if } \gamma\neq 0,\\ C_p(1+t)^{p/2} & \text{if } \gamma=0,p=2 \text{ or } \gamma=0,\gamma_2>0,p\in(2,p_0],\\ C_p(1+t)^p & \text{if } \gamma=0,\gamma_2=0,p\in(2,p_0],\end{cases}\tag{3.2}$$

where $\gamma=\gamma_1+\gamma_2$.

Note that if $\gamma<0$, we have that $\sup_{t\geq 0}\mathbb{E}\left[|X_t|^p\right]\leq 2C_p$.

3.3 Propagation of chaos

For $N \in \mathbb{N}$, suppose that (W^i, Z^i) are independent copies of the couple (W, Z) for $i \in \{1, \dots, N\}$. Let $N^i(dt, dz)$ be the Poisson random measure associated with the jumps of the Lévy process Z^i with intensity measure $\nu(dz)dt$, and $\widetilde{N}^i(dt, dz) := N^i(dt, dz) - \nu(dz)dt$ be the compensated Poisson random measure associated with $N^i(dt, dz)$. Thus, the Lévy-Itô decomposition of Z^i is given by $Z^i_t = \int_0^t \int_{\mathbb{R}^d_0} z\widetilde{N}^i(ds, dz)$ for $t \geq 0$. We now consider the system of non-interacting particles, which is associated with the Lévy-driven McKean-Vlasov SDE (3.1), where the state $X^i = (X^i_t)_{t\geq 0}$ of particle i is defined by

$$
\begin{aligned}
X^i_t &= x_0 + \int_0^t b(X^i_s, \mathcal{L}_{X^i_s})ds + \int_0^t \sigma(X^i_s, \mathcal{L}_{X^i_s})dW^i_s + \int_0^t c\left(X^i_{s-}, \mathcal{L}_{X^i_{s-}}\right) dZ^i_s \\
&= x_0 + \int_0^t b(X^i_s, \mathcal{L}_{X^i_s})ds + \int_0^t \sigma(X^i_s, \mathcal{L}_{X^i_s})dW^i_s \\
&\quad + \int_0^t \int_{\mathbb{R}^d_0} c\left(X^i_{s-}, \mathcal{L}_{X^i_{s-}}\right) z\widetilde{N}^i(ds, dz),
\end{aligned}
\tag{3.3}
$$

for any $t \geq 0$ and $i \in \{1, \dots, N\}$.

For $\boldsymbol{x}^N := (x_1, x_2, \dots, x_N), \boldsymbol{y}^N := (y_1, y_2, \dots, y_N) \in \mathbb{R}^{dN}$, we have

$$
\mathcal{W}_2^2(\mu^{\boldsymbol{x}^N}, \delta_0) = \frac{1}{N}\sum_{i=1}^N |x_i|^2 .
$$

Here, the empirical measure is defined by $\mu^{\boldsymbol{x}^N}(dx) := \frac{1}{N}\sum_{i=1}^N \delta_{x_i}(dx)$, where δ_x denotes the Dirac measure at x. Moreover, a standard bound for the Wasserstein distance between two empirical measures $\mu^{\boldsymbol{x}^N}, \mu^{\boldsymbol{y}^N}$ is given by

$$
\mathcal{W}_2^2(\mu^{\boldsymbol{x}^N}, \mu^{\boldsymbol{y}^N}) \leq \frac{1}{N}\sum_{i=1}^N |x_i - y_i|^2 = \frac{1}{N}\left|\boldsymbol{x}^N - \boldsymbol{y}^N\right|^2,
$$

(see (1.24) of [6]).

Now, the true measure $\mathcal{L}_{X_t}$ at each time t is approximated by the empirical measure

$$
\mu_t^{\boldsymbol{X}^N}(dx) := \frac{1}{N}\sum_{i=1}^N \delta_{X_t^{i,N}}(dx),
\tag{3.4}
$$

where $\boldsymbol{X}^N = (\boldsymbol{X}^N_t)_{t\geq 0} = (X_t^{1,N}, \dots, X_t^{N,N})^\intercal_{t\geq 0}$, which is called the system of interacting particles, is the solution to the $\mathbb{R}^{dN}$-dimensional Lévy-driven SDE with components $X^{i,N} = (X^{i,N}_t)_{t\geq 0}$

$$
\begin{aligned}
X_t^{i,N} &= x_0 + \int_0^t b(X_s^{i,N}, \mu_s^{\boldsymbol{X}^N})ds + \int_0^t \sigma(X_s^{i,N}, \mu_s^{\boldsymbol{X}^N})dW^i_s + \int_0^t c\left(X_{s-}^{i,N}, \mu_{s-}^{\boldsymbol{X}^N}\right) dZ^i_s \\
&= x_0 + \int_0^t b(X_s^{i,N}, \mu_s^{\boldsymbol{X}^N})ds + \int_0^t \sigma(X_s^{i,N}, \mu_s^{\boldsymbol{X}^N})dW^i_s
\end{aligned}
$$

$$+\int_0^t\int_{\mathbb{R}_0^d} c\left(X_{s-}^{i,N},\mu_{s-}^{\boldsymbol{X}^N}\right) z\widetilde{N}^i(ds,dz), \tag{3.5}$$

for any $t\geq 0$ and $i\in\{1,\ldots,N\}$.

Observe that the interacting particle system $\boldsymbol{X}^N=(X^{i,N})^{\mathsf{T}}_{i\in\{1,\ldots,N\}}$ can be viewed as an ordinary Lévy-driven SDE with random coefficients taking values in $\mathbb{R}^{dN}$. Therefore, under Conditions **C1**, **C3** and **C2** valid for $\kappa_1=\kappa_2=1, L_1=L_2>0$, there exists a unique càdlàg solution such that

$$\max_{i\in\{1,\ldots,N\}}\sup_{t\in[0,T]}\mathbb{E}\left[|X_t^{i,N}|^2\right]\leq K,$$

for any $N\in\mathbb{N}$, where $K>0$ does not depend on N.

Proposition 3.4.1. *Let $X^{i,N}=(X_t^{i,N})_{t\geq 0}$ be a solution to equation (3.5). Assume Conditions **C6, C7** hold and that σ is bounded on $C\times\mathcal{P}_2(\mathbb{R}^d)$ for every compact subset C of $\mathbb{R}^d$, and **C5** holds for $q=2p_0$. Then for any $p\in[2;p_0]$, there exists a positive constant C_p such that for any $t\geq 0$,*

$$\max_{i\in\{1,\ldots,N\}}\mathbb{E}\left[|X_t^{i,N}|^p\right]\leq\begin{cases}C_p(1+e^{\gamma pt}) & \text{if}\quad \gamma\neq 0,\\ C_p(1+t)^{p/2} & \text{if}\quad \gamma=0,p=2\quad\text{or}\quad\gamma=0,\gamma_2>0,p\in(2,p_0],\\ C_p(1+t)^p & \text{if}\quad\gamma=0,\gamma_2=0,p\in(2,p_0],\end{cases}$$

where $\gamma=\gamma_1+\gamma_2$.

Note that when $\gamma<0$, we have that $\max_{i\in\{1,\ldots,N\}}\sup_{t\geq 0}\mathbb{E}\left[|X_t^{i,N}|^p\right]\leq 2C_p$.

Next, we provide a result on the propagation of chaos which is the key to the convergence as $N\uparrow\infty$. To simplify the exposition, we define

$$\varphi(N)=\begin{cases}N^{-1/2} & \text{if}\quad d<4,\\ N^{-1/2}\ln N & \text{if}\quad d=4,\\ N^{-2/d} & \text{if}\quad d>4.\end{cases}$$

Proposition 3.4.2. *Assume that all conditions in Proposition 3.4.1 hold and that Condition **C2** holds for $\kappa_1=\kappa_2=1$, $L_1\in\mathbb{R}$, $L_2\geq 0$. Then, we have*

$$\max_{i\in\{1,\ldots,N\}}\sup_{t\in[0,T]}\mathbb{E}\left[\left|X_t^i-X_t^{i,N}\right|^2\right]\leq C_T\varphi(N), \tag{3.6}$$

for any $N\in\mathbb{N}$, where the positive constant C_T does not depend on N.

Assume further that $L_1+L_2<0$ and $\gamma<0$. Then, we have

$$\max_{i\in\{1,\ldots,N\}}\sup_{t\geq 0}\mathbb{E}\left[\left|X_t^i-X_t^{i,N}\right|^2\right]\leq C\varphi(N), \tag{3.7}$$

for any $N\in\mathbb{N}$, where the positive constant C does not depend on N and T.

3.4 Tamed-adaptive Euler-Maruyama scheme

Let $\sigma_\Delta = (\sigma_{\Delta,ij})_{1\le i,j\le d} : \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}^d \otimes \mathbb{R}^d$ and $c_\Delta = (c_{\Delta,ij})_{1\le i,j\le d} : \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}^d \otimes \mathbb{R}^d$ be approximations of the coefficients σ and c, respectively, which will be specified later. For all $i \in \{1,\dots,N\}$, $\Delta \in (0,1)$ and $k \in \mathbb{N}$, we define the tamed-adaptive Euler-Maruyama discretization of equation (3.5) by

$$\begin{cases} t_0 = 0, \quad \widehat{X}_0^{i,N} = x_0, \quad t_{k+1} = t_k + \mathbf{h}(\widehat{\boldsymbol{X}}_{t_k}^N, \mu_{t_k}^{\widehat{\boldsymbol{X}}^N})\Delta, \\ \widehat{X}_{t_{k+1}}^{i,N} = \widehat{X}_{t_k}^{i,N} + b(\widehat{X}_{t_k}^{i,N}, \mu_{t_k}^{\widehat{\boldsymbol{X}}^N})(t_{k+1}-t_k) + \sigma_\Delta(\widehat{X}_{t_k}^{i,N}, \mu_{t_k}^{\widehat{\boldsymbol{X}}^N})(W_{t_{k+1}}^i - W_{t_k}^i) \\ \qquad + c_\Delta(\widehat{X}_{t_k}^{i,N}, \mu_{t_k}^{\widehat{\boldsymbol{X}}^N})(Z_{t_{k+1}}^i - Z_{t_k}^i), \end{cases} \tag{3.8}$$

where

$$\widehat{\boldsymbol{X}}_{t_k}^N = \left(\widehat{X}_{t_k}^{1,N}, \dots, \widehat{X}_{t_k}^{N,N}\right),$$

$$\mu_{t_k}^{\widehat{\boldsymbol{X}}^N}(dx) := \frac{1}{N}\sum_{i=1}^N \delta_{\widehat{X}_{t_k}^{i,N}}(dx),$$

$$\mathbf{h}(\widehat{\boldsymbol{X}}_{t_k}^N, \mu_{t_k}^{\widehat{\boldsymbol{X}}^N}) = \min\left\{h(\widehat{X}_{t_k}^{1,N}, \mu_{t_k}^{\widehat{\boldsymbol{X}}^N}), \dots, h(\widehat{X}_{t_k}^{N,N}, \mu_{t_k}^{\widehat{\boldsymbol{X}}^N})\right\},$$

and

$$h(x,\mu) = \frac{h_0}{(1+|b(x,\mu)|+|\sigma(x,\mu)|+|x|^\ell)^2 + |c(x,\mu)|^{p_0}}, \tag{3.9}$$

for $x \in \mathbb{R}^d$, $\mu \in \mathcal{P}_2(\mathbb{R}^d)$ and some positive constant h_0. Here, the constants ℓ and p_0 are respectively defined in Conditions **C4** and **C7**.

Now, we provide sufficient conditions to ensure $t_k \uparrow \infty$ as $k \uparrow \infty$, which shows that the tamed-adaptive Euler-Maruyama approximation scheme (3.8) is well-defined.

Proposition 3.5.1. *Assume that Condition **C5** holds for $p = 2$ and there exist positive constants L, β_1 and β_2 such that the functions h, b, σ_Δ and c_Δ satisfy*

T1. $\frac{1}{h(x,\mu)} \le L\left(1 + |x|^{\beta_1} + \mathcal{W}_2^{\beta_2}(\mu,\delta_0)\right)$; $|b(x,\mu)|\,(1+|b(x,\mu)|)\,h(x,\mu) \le L$;

T2. $\langle x, b(x,\mu) - b(0,\delta_0)\rangle \le L\left(|x|^2 + \mathcal{W}_2^2(\mu,\delta_0)\right)$;

T3. $|\sigma_\Delta(x,\mu)|\,(1+|x|) \le \frac{L}{\sqrt{\Delta}}$; $|c_\Delta(x,\mu)|\,(1+|x|) \le \frac{L}{\sqrt{\Delta}}$;
$|b(x,\mu)||c_\Delta(x,\mu)| \le \frac{L}{\sqrt{\Delta}}$;

for any $x \in \mathbb{R}^d$ and $\mu \in \mathcal{P}_2(\mathbb{R}^d)$. Then, we have

$$\lim_{k\to+\infty} t_k = +\infty \quad a.s.$$

3.5 Moments

Firstly, we are going to show that the moments of $\widehat{X}_t^{i,N}$ depend on t. For this, we need to introduce the following condition.

T4. There exists a positive constant L such that $|\sigma_\Delta(x,\mu)| \leq |\sigma(x,\mu)|$ and $|c_\Delta(x,\mu)| \leq |c(x,\mu)|$ for all $x \in \mathbb{R}^d$ and $\mu \in \mathcal{P}_2(\mathbb{R}^d)$;

T5. For some integer $p_0 \in [2;+\infty)$, there exist constants $\widetilde{L_3} > 0$, $\widetilde{\gamma}_1 \in \mathbb{R}$, $\widetilde{\gamma}_2 > 0$, $\widetilde{\eta} \geq 0$ such that for all $x \in \mathbb{R}^d$ and $\mu \in \mathcal{P}_2(\mathbb{R}^d)$,

$$|c_\Delta(x,\mu)| \leq \widetilde{L_3}\left(1 + |x| + \mathcal{W}_2(\mu,\delta_0)\right), \tag{3.10}$$

and

$$\begin{aligned}
&\langle x, b(x,\mu)\rangle + \frac{p_0-1}{2}|\sigma_\Delta(x,\mu)|^2 + |c_\Delta(x,\mu)|^2 \int_{\mathbb{R}_0^d} \Bigg[\frac{|z|^2}{2} + \frac{1}{\widetilde{L_0}^2} \\
&\times \left(\left(1 + |z|(\widetilde{L_3}+\epsilon)\right)^{p_0-1} - 1 - |z|(\widetilde{L_3}+\epsilon)\right)\left(|z|\left(\frac{\widetilde{L_3}}{2}+\epsilon\right)+\epsilon\right)\Bigg]\nu(dz) \\
&\leq \widetilde{\gamma}_1|x|^2 + \widetilde{\gamma}_2\mathcal{W}_2^2(\mu,\delta_0) + \widetilde{\eta},
\end{aligned} \tag{3.11}$$

where $\epsilon = \frac{1}{2\sqrt{N}}\max\{3\widetilde{L_3}, 1\}$.

Proposition 3.6.9. *Assume Conditions **T1**–**T5** and **C5** hold. Then, for any positive $k \leq p_0/2$, there exists a positive constant $C = C(x_0, k, \widetilde{\gamma}_1, \widetilde{\gamma}_2, \widetilde{\eta}, L, \widetilde{L_3}, p_0)$ which depends neither on Δ nor on t such that for any $t \geq 0$,*

$$\max_{i\in\{1,\ldots,N\}} \left(\mathbb{E}\left[|\widehat{X}_t^{i,N}|^{2k}\right] \vee \mathbb{E}\left[|\widehat{X}_{\underline{t}}^{i,N}|^{2k}\right]\right) \leq \begin{cases} Ce^{2k\widetilde{\gamma}t} & \text{if } \widetilde{\gamma} > 0, \\ C(1+t)^k & \text{if } \widetilde{\gamma} = 0, \\ C & \text{if } \widetilde{\gamma} < 0, \end{cases} \tag{3.12}$$

where $\widetilde{\gamma} = \widetilde{\gamma}_1 + \widetilde{\gamma}_2$.

3.6 Convergence

Firstly, the following additional condition will be needed.

T6. There exists a positive constant L_4 such that

$$\begin{aligned}
|\sigma(x,\mu) - \sigma_\Delta(x,\mu)| &\leq L_3\Delta^{1/2}|\sigma(x,\mu)|^2(1+|x|), \\
|c(x,\mu) - c_\Delta(x,\mu)| &\leq L_4\Delta^{1/2}|c(x,\mu)|^2\left(1+|x|+|b(x,\mu)|\right),
\end{aligned}$$

for all $x \in \mathbb{R}^d$ and $\mu \in \mathcal{P}_2(\mathbb{R}^d)$.

Remark 3.7.2. If we choose

$$\sigma_\Delta(x,\mu) = \frac{\sigma(x,\mu)}{1+\Delta^{1/2}|\sigma(x,\mu)|(1+|x|)}, \tag{3.13}$$

$$c_\Delta(x,\mu) = \frac{c(x,\mu)}{1+\Delta^{1/2}|c(x,\mu)|(1+|x|+|b(x,\mu)|)}, \tag{3.14}$$

then Conditions **T3, T4** and **T6** are satisfied.

Theorem 3.7.3. *Assume that the coefficients $b, \sigma, c, \sigma_\Delta, c_\Delta$ and the Lévy measure ν satisfy Conditions **C1**, **C3–C5**, **T2–T6**, and $p_0 \geq 4\ell + 6$, $N \geq \left(\frac{\max\{3\widetilde{L_3},1\}}{2\epsilon}\right)^2$. Assume further that there exists a constant $\varepsilon > 0$ such that **C2** holds for $\kappa_1 = \kappa_2 = 1+\varepsilon$, $L_1 \in \mathbb{R}$, $L_2 \geq 0$. Then for any $T > 0$, there exist positive constants $C_T = C(x_0, L, L_1, L_2, L_4, \widetilde{\gamma}_1, \widetilde{\gamma}_2, \widetilde{\eta}, \widetilde{L_3}, \varepsilon, T)$ and $C'_T = C'(x_0, L, L_1, L_2, L_4, \widetilde{\gamma}_1, \widetilde{\gamma}_2, \widetilde{\eta}, \widetilde{L_3}, \varepsilon, T)$ such that*

$$\max_{i\in\{1,\dots,N\}} \sup_{t\in[0,T]} \mathbb{E}\left[\left|X_t^{i,N} - \widehat{X}_t^{i,N}\right|^2\right] \leq C_T\Delta, \tag{3.15}$$

and for any $p \in (0,2)$,

$$\max_{i\in\{1,\dots,N\}} \mathbb{E}\left[\sup_{t\in[0,T]}\left|X_t^{i,N} - \widehat{X}_t^{i,N}\right|^p\right] \leq \left(\frac{4-p}{2-p}\right)(C'_T\Delta)^{p/2}. \tag{3.16}$$

Moreover, if $\widetilde{\gamma} = \widetilde{\gamma}_1 + \widetilde{\gamma}_2 < 0$, and $L_1 + L_2 < 0$, then, there exists a positive constant $C'' = C''(x_0, L, L_1, L_2, L_4, \widetilde{\gamma}_1, \widetilde{\gamma}_2, \widetilde{\eta}, \widetilde{L_3}, \varepsilon)$, which does not depend on T, such that

$$\max_{i\in\{1,\dots,N\}} \sup_{t\geq 0} \mathbb{E}\left[\left|X_t^{i,N} - \widehat{X}_t^{i,N}\right|^2\right] \leq C''\Delta. \tag{3.17}$$

Theorem 3.7.4. *Assume Conditions **C1**, **C3–C7**, **T2–T6** hold, and $p_0 \geq 4\ell + 6$. Assume further that there exists a constant $\varepsilon > 0$ such that **C2** holds for $\kappa_1 = \kappa_2 = 1+\varepsilon$, $L_1 \in \mathbb{R}$, $L_2 \geq 0$. Then for any $T > 0$, there exists a positive constant $C_T = C(x_0, L, L_1, L_2, L_3, L_4, \widetilde{\gamma}_1, \widetilde{\gamma}_2, \widetilde{\eta}, \widetilde{L_3}, T)$ such that*

$$\max_{i\in\{1,\dots,N\}} \sup_{t\in[0,T]} \mathbb{E}\left[\left|X_t^{i} - \widehat{X}_t^{i,N}\right|^2\right] \leq C_T\left(\Delta + \varphi(N)\right), \tag{3.18}$$

for any $N \in \mathbb{N}$, where the constant $C_T > 0$ does not depend on N.

Moreover, assume that $\gamma = \gamma_1 + \gamma_2 < 0$, $\widetilde{\gamma} = \widetilde{\gamma}_1 + \widetilde{\gamma}_2 < 0$ and $L_1 + L_2 < 0$. Then, there exists a positive constant
$C'' = C''(x_0, L, L_1, L_2, L_3, L_4, \widetilde{\gamma}_1, \widetilde{\gamma}_2, \widetilde{\eta}, \widetilde{L_3})$ which does not depend on T such that

$$\max_{i\in\{1,\dots,N\}} \sup_{t\geq 0} \mathbb{E}\left[\left|X_t^{i} - \widehat{X}_t^{i,N}\right|^2\right] \leq C''\left(\Delta + \varphi(N)\right). \tag{3.19}$$

3.7 Numerical experiments

In this section, we consider the rate of convergence of the tamed-adaptive Euler-Maruyama scheme (3.8),(3.9), (3.13), (3.14) in Theorem 3.7.3 for fixed large values of N. We consider the following Lévy-driven McKean-Vlasov stochastic differential equation

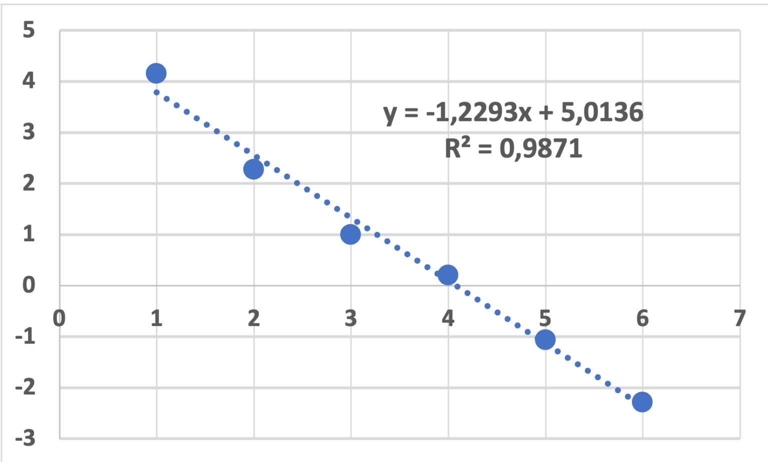
$$dX_t = \left(-1 - 3\left(X_t + \mathbb{E}\left[X_t\right]\right) - X_t|X_t|^{0.3}\right)dt + 0.2\left(1 + |X_t|^{1.1} + \mathbb{E}\left[X_t\right]\right)dW_t \\ + 0.2\left(X_{t-} + \mathbb{E}\left[X_{t-}\right]\right)dZ_t. \tag{3.20}$$

That is,

$$b(x,\mu) = -1 - 3\left(x + \int_{\mathbb{R}} z\mu(dz)\right) - x|x|^{0.3},$$
$$\sigma(x,\mu) = 0.2\left(1 + |x|^{1.1} + \int_{\mathbb{R}} z\mu(dz)\right), \qquad c(x,\mu) = 0.2\left(x + \int_{\mathbb{R}} z\mu(dz)\right),$$

for all $x \in \mathbb{R}$ and $\mu \in \mathcal{P}_2(\mathbb{R})$. Here we suppose that $Z = (Z_t)_{t\geq 0}$ is a bilateral Gamma process whose scale parameter is 5 and shape parameter is 1. It is straightforward to verify that these coefficients satisfy Conditions **C1–C7** and **T2–T6**.

Figure 3.1 shows the values of $\log_2 MSE(\mathbf{l}, T)$ plotted against $\mathbf{l} \in \{1, 2, ..., 6\}$. We see that $\beta \approx 0.5$.



Hình 3.1: Error $\log_2\text{MSE}(\mathbf{l}, 10)$ plotted against $\mathbf{l} = 1, ..., 6$.

CONCLUSIONS AND RECOMMENDATIONS

Conclusions

This thesis leverages modern tools from stochastic analysis, stochastic differential equations (SDEs), numerical analysis, and Yamada-Watanabe approximation techniques to construct appropriate approximation schemes for several classes of SDEs with irregular coefficients. The central results focus on the quantitative and qualitative properties of the exact solutions and the approximated solutions (via the Euler-Maruyama scheme) for SDEs with irregular coefficients and for SDEs with jumps. The thesis proposes a tamed-adaptive Euler-Maruyama approximation scheme and establishes its strong convergence, over both finite and infinite time intervals, for the following three classes of equations:

- **Classical SDEs:** Applied to a class of SDEs where the drift coefficient is locally Lipschitz continuous and the diffusion coefficient is locally Hölder continuous.
- **Lévy-driven SDEs:** Applied to Lévy-driven SDEs where b (drift) is locally Lipschitz continuous, σ (diffusion) is locally Hölder continuous, and c (jump) is Lipschitz continuous.
- **Lévy-driven McKean-Vlasov SDEs:** Applied to Lévy-driven McKean-Vlasov SDEs where b, σ and c are non-globally Lipschitz continuous and exhibit super-linear growth.

Recommendations

Based on the research conducted in this thesis, we identify the following promising directions for future work:

- Developing and analyzing approximation methods that preserve structural properties. This includes preserving the geometrical or asymptotic properties of stochastic differential equations (SDEs) with complex structures, such as systems of non-colliding random points or positivity-preserving systems.
- Investigating the weak convergence of the tamed-adaptive schemes. The current work has focused on strong convergence; analyzing the weak convergence properties and rates would be a valuable extension.
- Constructing higher-order approximation schemes. This involves developing schemes with faster convergence rates for SDEs that have smooth coefficients but still exhibit super-linear growth.

LIST OF AUTHOR'S RELATED PAPERS

[1] Kieu T.T., Luong D.T., Ngo H.L. (2022), "Tamed-adaptive Euler-Maruyama approximation for SDEs with locally Lipschitz continuous drift and locally Hölder continuous diffusion coefficients", *Stoch Anal Appl* 40(4), pp. 714-734.

[2] Kieu T.T., Luong D.T., Ngo H.L., Tran N.K. (2022), "Strong convergence in infinite time interval of tamed-adaptive Euler-Maruyama scheme for Lévy-driven SDEs with irregular coefficients", *Comp. Appl. Math.* 41, 301.

[3] Tran N.K., Kieu T.T., Luong D.T., Ngo H.L. (2025), "On the infinite time horizon approximation for Lévy-driven McKean-Vlasov SDEs with non-globally Lipschitz continuous and super-linearly growth drift and diffusion coefficients", *J. Math. Anal. Appl.*, 54, paper no. 128982, 38 pp.